

Krichever correspondence for algebraic varieties

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Abstract

In the work is constructed new acyclic resolutions of quasicoherent sheaves. These resolutions is connected with multidimensional local fields. Then the obtained resolutions is applied for a construction of generalization of the Krichever map to algebraic varieties of any dimension.

This map gives in the canonical way two k -subspaces $B \subset k((z_1)) \dots ((z_n))$ and $W \subset k((z_1)) \dots ((z_n))^{\oplus r}$ from arbitrary algebraic n -dimensional Cohen-Macaulay projective integral scheme X over a field k , a flag of closed integral subschemes $X = Y_0 \supset Y_1 \supset \dots \supset Y_n$ (such that Y_i is an ample Cartier divisor on Y_{i-1} , and Y_n is a smooth k -point on all Y_i), formal local parameters of this flag in the point Y_n , a rank r vector bundle \mathcal{F} on X , and a trivialization \mathcal{F} in the formal neighbourhood of the point Y_n , where the n -dimensional local field $k((z_1)) \dots ((z_n))$ is associated with the flag $Y_0 \supset \dots \supset Y_n$. In addition, the constructed map is injective, i. e., it is possible to reconstruct uniquely all the original geometrical data. Besides, from the subspace B is written explicitly a complex, which calculates cohomology of the sheaf \mathcal{O}_X on X ; and from the subspace W is written explicitly a complex, which calculates cohomology of \mathcal{F} on X .

1 Introduction

In 70's years I. M. Krichever suggested a construction how to attach to some algebraic-geometric data, connected with algebraic curves and vector bundles on them, an infinite-dimensional (Fredholm) subspace in the space $k((z))$ of Laurent power series ([12]). This construction was successfully used in the theory of integrable systems, in particular, in the theory of KP and KdV equations ([12], [19], [6]).

There were also found applications of this construction to the theory of modules of algebraic curves ([1], [4]). Besides, this construction turned out to be connected with description of commutative subrings in the rings of pseudo-differential operators ([13], [6]). Now this construction is called the Krichever correspondence or the Krichever map ([1], [13], [5], [18]). But in these works it is essentially that algebraic-geometric data are connected with 1-dimensional varieties and 1-dimensional local field $k((z))$.

Recently, in works [16], [17] it were pointed out some connections between the theory of the KP-equations and n -dimensional local fields; also it was suggested a variant of the Krichever map for algebraic-geometric data which is connected with algebraic surfaces, vector bundles on them and 2-dimensional local fields.

One of the typical examples of multidimensional local field is the field of Laurent iterated series $k((z_1)) \dots ((z_n))$.

Such fields serve for natural generalization of local objects of 1-dimensional varieties to the case of multidimensional varieties. Let us consider an n -dimensional algebraic scheme X . Let $Y_0 \supset \dots \supset Y_n$ be a flag of closed subschemes on X such that $Y_0 = X$, Y_i is of codimension 1 in Y_{i-1} , and $Y_n = x$ is a closed point. Then there exists a construction ([14], [3], [15]), attaching in the canonical way to such flag some ring, which is an n -dimensional local field provided that x is a smooth point on all Y_i . Moreover, if X is an algebraic variety over a field k , x is a k -rational point, and we fix local parameters $z_1, z_2, \dots, z_n \in \hat{\mathcal{O}}_{x,X}$ such that $z_{n-i+1} = 0$ is a local equation of variety Y_i in the formal neighbourhood of the point x on the variety Y_{i-1} ($1 \leq i \leq n$), then the obtained n -dimensional local field it is possible to identify with $k((z_1))((z_2)) \dots ((z_n))$.

A concept of multidimensional local field has appeared in the middle of 70's years, and, originally, such fields were used for the development of generalization of class field theory to the schemes of higher dimension. Later there were also found applications of multidimensional local fields to many problems of algebraic geometry, where it makes sense to speak about local components of geometric objects (see [7]).

In this work, using multidimensional local fields, we construct the Krichever correspondence for varieties of arbitrary dimension n : that is some injective map from algebraic-geometric data, connected with projective algebraic varieties, full flags of ample divisors and their local parameters in the formal neighbourhood of the last point of the flag, vector bundles and their trivialisations in the formal neighbourhood of the last point of the flag, to some k -subspaces of finite dimensional vector space over the n -dimensional local field $k((z_1)) \dots ((z_n))$.

If $n = 1$, then our constructed map is a variant of the Krichever map for curves.

If $n = 2$, then our constructed map coincides with the map constructed in [17].

The work is organized as follows.

In §2 we give various technical lemmas about cohomology of coherent sheaves, projective and injective limits, which will be useful further in the work.

In §3 we give a construction of family of functors, which is connected with quasicoherent sheaves and a fixed flag of subvarieties, and which can be interpreted as a cohomology system of coefficients on the standart simplex.

In §4, using the construction of §3, we construct complexes of sheaves of abelian groups, which is acyclic resolutions of arbitrary quasicoherent sheaves on schemes.

In §5 we prove some theorems about intersections among components of resolutions constructed in §4. In some cases the whole resolution can be reconstructed from one k -subspace of finite dimensional vector space over $k((z_1)) \dots ((z_n))$. Using this, we construct the Krichever map in higher dimensions.

Note that connected with n -dimensional local fields resolutions of quasicoherent sheaves on schemes were in works [3], [11]. But in contrast to these works, our resolutions depend only on a single flag of subvarieties and are not resolutions of adelic type.

Note also that, in contrast to [17], all the constructions and proofs in this work are internal ones, i. e., they are not reduced to multidimensional adelic complexes.

During all the work we shall keep the following notations and agreements.

For any finite set I let $\sharp I$ be the number of elements of the set I .

If X is a scheme, then

$Sh(X)$ is the category of sheaves of abelian groups on X ,

$CS(X)$ is the category of coherent sheaves on X ,

$QS(X)$ is the category of quasicoherent sheaves on X ,

Ab is the category of abelian groups.

If $f : Y \longrightarrow X$ is a morphism of two schemes, then always f^* is the pull-back functor in the category of sheaves of abelian groups, f_* is the direct image functor in the category of sheaves of abelian groups.

If \mathcal{U} is an open covering of X , \mathcal{F} is a sheaf of abelian groups on X , then $\check{H}^*(\mathcal{U}, \mathcal{F})$ are the Čech cohomologies groups with respect to the covering \mathcal{U} .

Let $Y \hookrightarrow X$ be a closed subscheme of a scheme X , which is defined by the ideal sheaf J . Then by $(Y, \mathcal{O}_X/J^k)$ denote the scheme whose topological space coincides with the topological space of the scheme Y and the structure sheaf is \mathcal{O}_X/J^k . (\mathcal{O}_X is the structure sheaf of the scheme X .)

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2 Technical lemmas

Lemma 1 *On a noetherian scheme X any short exact sequence of quasicoherent sheaves is direct limit of short exact sequences of coherent sheaves. For $\phi : \mathcal{F} \longrightarrow \mathcal{G} \in \text{Mor}(QS(X))$ there are $\phi_i : \mathcal{F}_i \longrightarrow \mathcal{G}_i \in \text{Mor}(CS(X))$ with $\varinjlim \phi_i = \phi$, $\mathcal{F}_i \subset \mathcal{F}$, $\mathcal{G}_i \subset \mathcal{G}$.*

Proof. See [11, lemma 1.2.2] and [10, lemma 2.1.5].

Lemma 2 *Let X be a noetherian scheme. Let $\psi : CS(X) \longrightarrow Sh(X)$ be an exact additive functor. Then ψ commutes with direct limits.*

Proof. (By analogy with [11, lemma 1.2.3] or [10, lemma 2.2.2].)

First let us prove that if we have a direct system of sheaves

$$\{\mathcal{F}_i : i \in I, \phi_{ij} : \mathcal{F}_i \rightarrow \mathcal{F}_j (i \leq j)\}$$

with $\varinjlim \mathcal{F}_i = 0$, then $\varinjlim \psi(\mathcal{F}_i) = 0$.

For this one we prove that for any open $U \subset X$ $\varinjlim H^0(U, \psi(\mathcal{F}_i)) = 0$. Let $x \in \varinjlim H^0(U, \psi(\mathcal{F}_i))$. Let this x be represented by $x_i \in H^0(U, \psi(\mathcal{F}_i))$. Then from coherent property of the sheaf \mathcal{F}_i and noetherian property of the scheme X there is some $j \in I$ such that $\phi_{ij} = 0$. Since ψ is an additive functor, we have that $\psi(\phi_{ij}) = 0$. Therefore

$$\begin{array}{ccc} H^0(U, \psi(\mathcal{F}_i)) & \longrightarrow & H^0(U, \psi(\phi_{ij})(\psi(\mathcal{F}_i))) \\ x_i & \mapsto & 0 \end{array}$$

Now consider the general case: let $\varinjlim \mathcal{F}_i = \mathcal{F}$, $\phi_i : \mathcal{F}_i \rightarrow \mathcal{F}$ be the canonical morphisms. Consider the following exact sequence of coherent sheaves:

$$0 \longrightarrow \text{Ker } \phi_i \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F} \longrightarrow \text{Coker } \phi_i \longrightarrow 0$$

The functor ψ is an exact functor, therefore we have the following exact sequence:

$$0 \longrightarrow \psi(\text{Ker } \phi_i) \longrightarrow \psi(\mathcal{F}_i) \longrightarrow \psi(\mathcal{F}) \longrightarrow \psi(\text{Coker } \phi_i) \longrightarrow 0$$

From $\varinjlim \text{Ker } \phi_i = 0$ and $\varinjlim \text{Coker } \phi_i = 0$ it follows by arguments above that $\varinjlim \psi(\text{Ker } \phi_i) = 0$ and $\varinjlim \psi(\text{Coker } \phi_i) = 0$. Direct limit maps exact sequences to exact sequence. Therefore $\varinjlim \psi(\mathcal{F}_i) = \psi(\mathcal{F})$. Lemma 2 is proved.

Lemma 3 *Let X be a noetherian scheme. Then an exact additive functor $\psi : CS(X) \longrightarrow Sh(X)$ can be uniquely extended to a functor $\psi' : QS(X) \longrightarrow Sh(X)$ which commutes with direct limits. This new functor is exact as well.*

Proof. (By analogy with [11, lemma 1.2.4].)

Let $\mathcal{F} \in \text{Ob}(QS(X))$. By lemma 1 $\mathcal{F} = \varinjlim \mathcal{F}_i$, where $\mathcal{F}_i \in \text{Ob}(CS(X))$. Define

$$\psi'(\mathcal{F}) = \varinjlim \psi(\mathcal{F}_i).$$

We have $\psi'(\mathcal{F}) = \psi(\mathcal{F})$ for $\mathcal{F} \in \text{Ob}(CS(X))$ by lemma 2. By lemma 1, for any $\phi \in \text{Mor}(QS(X))$ we have $\phi = \varinjlim \phi_i$, where $\phi_i \in \text{Mor}(CS(X))$. Define

$$\psi'(\phi) = \varinjlim \psi(\phi_i).$$

By lemma 2, we have that $\psi'(\phi) = \psi(\phi)$ for $\phi \in \text{Mor}(CS(X))$. It is clear that this definition is the only one possible. And by lemma 2, it is well defined. Lemma 3 is proved.

Lemma 4 *Let X be a noetherian scheme, $i : Y \hookrightarrow X$ be a closed subscheme, which is defined by the ideal sheaf J on X . Let $j : U \hookrightarrow Y$ be an open subscheme of Y such that for any point $x \in X$ there exists an affine neighborhood $V \ni x$ such that $V \cap U$ is an affine subscheme. Let the supports of sheaves $\mathcal{F}_i \in Sh(X)$ ($i = 1, \dots, 3$) are in Y , and the sheaf \mathcal{F}_1 is a quasicohherent sheaf with respect to the subscheme $(Y, \mathcal{O}_X/J^k)$ for some $k \in \mathbb{N}$. Then from exactness of the sequence of sheaves*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0 \tag{1}$$

it follows exactness of the following sequence

$$0 \longrightarrow i_* j_* j^* \mathcal{F}_1 \longrightarrow i_* j_* j^* \mathcal{F}_2 \longrightarrow i_* j_* j^* \mathcal{F}_3 \longrightarrow 0 \quad .$$

Proof. First note that for any affine open subscheme $W \subset U$ and for any quasicoherent sheaf \mathcal{G} on the scheme $(U, \mathcal{O}_X/J^k|_U)$ we have

$$H^1(W, \mathcal{G}) = 0 \quad . \quad (2)$$

In fact, if the sheaf \mathcal{G} is a quasicoherent sheaf with respect to the subscheme $U = (U, \mathcal{O}_X/J|_U)$, then equality (2) follows from affineness of the scheme W . Now if \mathcal{F} is a quasicoherent sheaf with respect to the subscheme $(U, \mathcal{O}_X/J^k|_U)$, $k \in \mathbf{N}$, $k \geq 1$, then consider the following exact sequence:

$$0 \longrightarrow J\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/J\mathcal{F} \longrightarrow 0 \quad . \quad (3)$$

But the sheaves $J\mathcal{F}$ and $\mathcal{F}/J\mathcal{F}$ are quasicoherent sheaves with respect to the subscheme $(U, \mathcal{O}_X/J^{k-1}|_U)$. Therefore we can do induction, from which it follows that

$$H^1(W, J\mathcal{F}) = 0 \quad \text{and} \quad H^1(W, \mathcal{F}/J\mathcal{F}) = 0 \quad .$$

Hence and from the long cohomological sequence associated with sequence (3) we obtain equality (2).

Return to sequence (1). We have exactness of the following sequence:

$$0 \longrightarrow j^*\mathcal{F}_1 \longrightarrow j^*\mathcal{F}_2 \longrightarrow j^*\mathcal{F}_3 \longrightarrow 0 \quad .$$

Applying the functor j_* , we obtain

$$0 \longrightarrow j_*j^*\mathcal{F}_1 \longrightarrow j_*j^*\mathcal{F}_2 \longrightarrow j_*j^*\mathcal{F}_3 \longrightarrow R^1j_*(j^*\mathcal{F}_1)$$

Let us show that the sheaf

$$R^1j_*(j^*\mathcal{F}_1) = 0 \quad .$$

The sheaf $j^*\mathcal{F}_1$ is a quasicoherent sheaf with respect to the subscheme $(U, \mathcal{O}_X/J^k|_U)$ for some $k \in \mathbf{N}$, therefore the sheaf $R^1j_*(j^*\mathcal{F}_1)$ is a quasicoherent sheaf on the scheme $(Y, \mathcal{O}_X/J^k)$ with respect to the same $k \in \mathbf{N}$. Therefore it suffices to show that for affine open V from the lemma's conditions

$$H^0(V \cap Y, R^1j_*(j^*\mathcal{F}_1)) = 0 \quad . \quad (4)$$

But $H^0(V \cap Y, R^1j_*(j^*\mathcal{F}_1)) = H^1(V \cap U, j^*\mathcal{F}_1)$. And equality (4) follows from equality (2). Therefore we have exactness of the following sequence:

$$0 \longrightarrow j_*j^*\mathcal{F}_1 \longrightarrow j_*j^*\mathcal{F}_2 \longrightarrow j_*j^*\mathcal{F}_3 \longrightarrow 0 \quad .$$

From $i : Y \hookrightarrow X$ is a closed imbedding it follows that i_* is an exact functor. Therefore the following sequence is exact:

$$0 \longrightarrow i_*j_*j^*\mathcal{F}_1 \longrightarrow i_*j_*j^*\mathcal{F}_2 \longrightarrow i_*j_*j^*\mathcal{F}_3 \longrightarrow 0 \quad .$$

Lemma 4 is proved.

Let X be a noetherian scheme. Suppose that we have an exact and additive functor $\Phi : QS(X) \longrightarrow Sh(X)$. Let $i : Y \hookrightarrow X$ be a closed subscheme of the scheme X , which is defined by the ideal sheaf J . Let $j : U \hookrightarrow Y$ be an open subscheme of Y . Then define a functor

$$C_U \Phi : CS(X) \longrightarrow Sh(X) \quad \text{as following:}$$

for any sheaf $\mathcal{F} \in CS(X)$:

$$C_U \Phi(\mathcal{F}) \stackrel{\text{def}}{=} \varprojlim_{k \in \mathbf{N}} \Phi(i_* j_* j^*(\mathcal{F}/J^k \mathcal{F})) \quad .$$

Remark 1 It is not difficult to understand that if the sheaf \mathcal{F} is a coherent sheaf on X , then for any $k \in \mathbf{N}$ the sheaf $i_* j_* j^*(\mathcal{F}/J^k \mathcal{F})$ is a quasicoherent sheaf on X . In fact, the sheaf $\mathcal{F}/J^k \mathcal{F}$ is a coherent sheaf on the scheme X . Moreover, the sheaf $\mathcal{F}/J^k \mathcal{F}$ is a coherent sheaf on the closed subscheme $(Y, \mathcal{O}_X/J^k)$ of the scheme X . Then $j^*(\mathcal{F}/J^k \mathcal{F})$ is a coherent sheaf on the scheme $(U, \mathcal{O}_X/J^k|_U)$, $j_* j^*(\mathcal{F}/J^k \mathcal{F})$ is a quasicoherent sheaf on the scheme $(Y, \mathcal{O}_X/J^k)$. And since i_* coincides with the direct image functor from the subscheme $(Y, \mathcal{O}_X/J^k)$, we see that $i_* j_* j^*(\mathcal{F}/J^k \mathcal{F})$ is a quasicoherent sheaf on X .

Lemma 5 *Let X be a noetherian scheme, $\Phi : QS(X) \longrightarrow Sh(X)$ be an exact additive functor, $i : Y \hookrightarrow X$ be a closed subscheme of the scheme X , which is defined by the ideal sheaf J on X . Let $j : U \hookrightarrow Y$ be an open subscheme of Y . In addition, suppose the following: for any point $x \in X$, for any open $W \subset X$, $x \in W$, there exists an affine open subscheme $V \subset W$, $x \in V$ such that:*

1. $V \cap U$ is an affine subscheme;
2. for any quasicoherent sheaf \mathcal{F} on X

$$H^1(V, \Phi(\mathcal{F})) = 0 \tag{5}$$

Then $C_U \Phi : CS(X) \longrightarrow Sh(X)$ is an exact and additive functor.

Remark 2 For example, condition 1 of lemma 5 is satisfied in the following cases:

- X is a separated scheme, U is an affine subscheme (as on a separated scheme the intersection of two affine open subschemes is an affine subscheme);
- X is a separated scheme, and U is a complement to some Cartier divisor in Y .

Proof (of lemma 5).

Additivity of the functor $C_U \Phi$ is obvious from the construction. Let us show exactness. Let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be an exact sequence of coherent sheaves on X .

For any $x \in X$, for any open $W \subset X$ consider an open affine $V \subset W$, $V \ni x$ satisfying conditions 1–2 of lemma 5. Then for the proof of exactness of the functor $C_U\Phi$ it suffices to show exactness of the following sequence:

$$0 \longrightarrow H^0(V, C_U\Phi(\mathcal{F}_1)) \longrightarrow H^0(V, C_U\Phi(\mathcal{F}_2)) \longrightarrow H^0(V, C_U\Phi(\mathcal{F}_3)) \longrightarrow 0. \quad (6)$$

On the other hand, the following sequence is exact:

$$0 \longrightarrow \mathcal{F}_1/J^k\mathcal{F}_2 \cap \mathcal{F}_1 \longrightarrow \mathcal{F}_2/J^k\mathcal{F}_2 \longrightarrow \mathcal{F}_3/J^k\mathcal{F}_3 \longrightarrow 0.$$

Since the sheaves in the last sequence are coherent sheaves on the scheme $(Y, \mathcal{O}_X/J^k)$, by lemma 4 the following sequence is exact:

$$0 \longrightarrow i_*j_*j^*(\mathcal{F}_1/J^k\mathcal{F}_2 \cap \mathcal{F}_1) \longrightarrow i_*j_*j^*(\mathcal{F}_2/J^k\mathcal{F}_2) \longrightarrow i_*j_*j^*(\mathcal{F}_3/J^k\mathcal{F}_3) \longrightarrow 0.$$

Since we apply the exact functor Φ to the last sequence, we obtain exactness of the following sequence from $Sh(X)$:

$$0 \longrightarrow \Phi(i_*j_*j^*(\mathcal{F}_1/J^k\mathcal{F}_2 \cap \mathcal{F}_1)) \longrightarrow \Phi(i_*j_*j^*(\mathcal{F}_2/J^k\mathcal{F}_2)) \longrightarrow \Phi(i_*j_*j^*(\mathcal{F}_3/J^k\mathcal{F}_3)) \longrightarrow 0.$$

Now from (5) we obtain exactness of the following sequence:

$$\begin{aligned} 0 \longrightarrow H^0(V, \Phi(i_*j_*j^*(\mathcal{F}_1/J^k\mathcal{F}_2 \cap \mathcal{F}_1))) &\longrightarrow H^0(V, \Phi(i_*j_*j^*(\mathcal{F}_2/J^k\mathcal{F}_2))) \longrightarrow \\ &\longrightarrow H^0(V, \Phi(i_*j_*j^*(\mathcal{F}_3/J^k\mathcal{F}_3))) \longrightarrow 0. \end{aligned} \quad (7)$$

Also for any natural numbers $k_1 \leq k_2$ we have the following exact sequence of sheaves:

$$0 \longrightarrow J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1 \longrightarrow \mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1 \longrightarrow \mathcal{F}_1/J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1 \longrightarrow 0.$$

Hence, as above, the sheaf $J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1$ is a coherent sheaf on the scheme $(Y, \mathcal{O}_X/J^{k_2})$. Therefore by lemma 4 the following sequence is exact:

$$\begin{aligned} 0 \longrightarrow i_*j_*j^*(J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1) &\longrightarrow i_*j_*j^*(\mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1) \longrightarrow \\ &\longrightarrow i_*j_*j^*(\mathcal{F}_1/J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1) \longrightarrow 0. \end{aligned}$$

Since the functor Φ is exact, we have exactness of the sequence:

$$\begin{aligned} 0 \longrightarrow \Phi(i_*j_*j^*(J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1)) &\longrightarrow \Phi(i_*j_*j^*(\mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1)) \longrightarrow \\ &\longrightarrow \Phi(i_*j_*j^*(\mathcal{F}_1/J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1)) \longrightarrow 0. \end{aligned}$$

And from (5) we obtain that the following map is a surjective map:

$$\Phi(i_*j_*j^*(\mathcal{F}_1/J^{k_2}\mathcal{F}_2 \cap \mathcal{F}_1)) \longrightarrow \Phi(i_*j_*j^*(\mathcal{F}_1/J^{k_1}\mathcal{F}_2 \cap \mathcal{F}_1)) \quad (8)$$

Now taking the projective limit with respect to all $k \in \mathbf{N}$ and using (8), from which it follows the Mittag-Leffler condition (see [9, ch.II, §9]), we obtain exactness of the following sequence:

$$\begin{aligned}
0 \rightarrow \varprojlim_{k \in \mathbf{N}} H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_1/J^k \mathcal{F}_2 \cap \mathcal{F}_1))) &\rightarrow \varprojlim_{k \in \mathbf{N}} H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_2/J^k \mathcal{F}_2))) \rightarrow \\
&\rightarrow \varprojlim_{k \in \mathbf{N}} H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_3/J^k \mathcal{F}_3))) \rightarrow 0
\end{aligned}$$

or

$$\begin{aligned}
0 \longrightarrow \varprojlim_{k \in \mathbf{N}} H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_1/J^k \mathcal{F}_2 \cap \mathcal{F}_1))) &\longrightarrow \\
&\longrightarrow H^0(V, C_U \Phi(\mathcal{F}_2)) \longrightarrow H^0(V, C_U \Phi(\mathcal{F}_3)) \longrightarrow 0.
\end{aligned}$$

Thus for the proof of exactness of sequence (6) we have to show that

$$H^0(V, C_U \Phi(\mathcal{F}_1)) = \varprojlim_{k \in \mathbf{N}} H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_1/J^k \mathcal{F}_2 \cap \mathcal{F}_1))). \quad (9)$$

Since the scheme X is a noetherian scheme, by Artin-Rees lemma (see [2, cor. 10.10]) there exists $l \in \mathbf{N}$ such that for all $k \leq l$:

$$J^k \mathcal{F}_2 \cap \mathcal{F}_1 = J^{k-l}(J^l \mathcal{F}_2 \cap \mathcal{F}_1). \quad (10)$$

From (10) we obtain that the maps

$$\mathcal{F}_1/J^k \mathcal{F}_2 \cap \mathcal{F}_1 \longrightarrow \mathcal{F}_1/J^{k-l} \mathcal{F}_1$$

are well defined and surjective. The same is for

$$\mathcal{F}_1/J^k \mathcal{F}_1 \longrightarrow \mathcal{F}_1/J^k \mathcal{F}_2 \cap \mathcal{F}_1 \quad .$$

Further, applying successively the functors j^* , j_* , i_* , Φ and $H^0(V, \cdot)$ and using lemma 4 and condition (5), we obtain cofinality of the projective systems:

$$H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_1/J^k \mathcal{F}_2 \cap \mathcal{F}_1))) \quad \text{and} \quad H^0(V, \Phi(i_* j_* j^*(\mathcal{F}_1/J^k \mathcal{F}_1))).$$

Therefore (9) is satisfied, and lemma 5 is proved.

Lemma 6 *Let X be a noetherian scheme, $\Phi : QS(X) \longrightarrow Sh(X)$ be an exact additive functor, $i : Y \hookrightarrow X$ be a closed subscheme of the scheme X , which is defined by the ideal sheaf J on X . Let $j : U \hookrightarrow Y$ be an open subscheme of Y such that for any point $x \in X$ there exists an affine neighbourhood $V \ni x$ such that $V \cap U$ is an affine subscheme. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of the scheme X . Suppose that for any $k \geq 1$, for any coherent sheaf \mathcal{G} on the scheme X we have*

$$\check{H}^m(\mathcal{U}, \Phi(i_* j_* j^*(\mathcal{G}/J^k \mathcal{G}))) = 0 \quad \text{for any } m \geq 1 \quad (11)$$

$$H^1(\bigcap_{i \in I_0} U_i, \Phi(i_* j_* j^*(\mathcal{G}/J^k \mathcal{G}))) = 0 \quad \text{for any subset } I_0 \subset I \quad (12)$$

$$H^1(X, \Phi(i_* j_* j^*(\mathcal{G}/J^k \mathcal{G}))) = 0 \quad . \quad (13)$$

Then $\check{H}^m(\mathcal{U}, C_U \Phi(\mathcal{F})) = 0$ for any $m \geq 1$ and for any coherent sheaf \mathcal{F} on X .

Proof. Let \mathcal{F} be any coherent sheaf on X . Let

$$p_n : \prod_{\substack{I_0 \subset I \\ \#I_0 = n+1}} H^0\left(\bigcap_{i \in I_0} U_i, \Phi(i_* j_* j^*(\mathcal{F}/J^k \mathcal{F}))\right) \longrightarrow \prod_{\substack{I_0 \subset I \\ \#I_0 = n+2}} H^0\left(\bigcap_{i \in I_0} U_i, \Phi(i_* j_* j^*(\mathcal{F}/J^k \mathcal{F}))\right)$$

be the map which is arised from the Čech complex with respect to the covering \mathcal{U} . Define $H_k^n \stackrel{\text{def}}{=} \text{Ker } p_n$. In addition,

$$H_k^0 = H^0(X, \Phi(i_* j_* j^*(\mathcal{F}/J^k \mathcal{F}))) \quad .$$

From (11) we obtain at once that for any $n \geq 1$

$$H_k^n = \text{Im } p_{n-1} \quad .$$

Therefore for any $m \geq 0$ the following sequence is exact:

$$0 \longrightarrow H_k^n \longrightarrow \prod_{\substack{I_0 \subset I \\ \#I_0 = n+1}} H^0\left(\bigcap_{i \in I_0} U_i, \Phi(i_* j_* j^*(\mathcal{F}/J^k \mathcal{F}))\right) \xrightarrow{p_n} H_k^{n+1} \longrightarrow 0 \quad (14)$$

For any natural numbers $k_1 \leq k_2$ we have the exact sequence

$$0 \longrightarrow J^{k_1} \mathcal{F} / J^{k_2} \mathcal{F} \longrightarrow \mathcal{F} / J^{k_2} \mathcal{F} \longrightarrow \mathcal{F} / J^{k_1} \mathcal{F} \longrightarrow 0.$$

Since the sheaves of this sequence are coherent sheaves on the scheme $(Y, \mathcal{O}_X/J^{k_2})$, by lemma 4 the following sequence is exact:

$$0 \longrightarrow i_* j_* j^*(J^{k_1} \mathcal{F} / J^{k_2} \mathcal{F}) \longrightarrow i_* j_* j^*(\mathcal{F} / J^{k_2} \mathcal{F}) \longrightarrow i_* j_* j^*(\mathcal{F} / J^{k_1} \mathcal{F}) \longrightarrow 0.$$

Further, from exactness of the functor Φ and condition (13) we obtain surjectivity of the following maps for any natural numbers $k_1 \leq k_2$:

$$H^0(X, \Phi(i_* j_* j^*(\mathcal{F} / J^{k_2} \mathcal{F}))) \longrightarrow H^0(X, \Phi(i_* j_* j^*(\mathcal{F} / J^{k_1} \mathcal{F}))). \quad (15)$$

By condition (12), we obtain as well that for any $k_1 \leq k_2$ and any $I_0 \subset I$ the maps

$$H^0\left(\bigcap_{i \in I_0} U_i, \Phi(i_* j_* j^*(\mathcal{F} / J^{k_2} \mathcal{F}))\right) \longrightarrow H^0\left(\bigcap_{i \in I_0} U_i, \Phi(i_* j_* j^*(\mathcal{F} / J^{k_1} \mathcal{F}))\right). \quad (16)$$

are surjective maps.

Let us prove that for any $n \geq 0$ the map

$$H_{k_2}^n \longrightarrow H_{k_1}^n \quad (17)$$

is a surjective map for any $k_1 \leq k_2$. In the case $n = 0$ it is statement (15). For arbitrary n it follows from surjectivity of p_{n-1} in exact sequence (14) and, also, surjectivity of map (16).

Now taking the projective limit in (14) and using surjectivity of (17), from which it follows the Mittag-Leffler condition for the projective systems n_k , we obtain exactness of the following sequence for any $n \geq 0$:

$$0 \longrightarrow \varprojlim_k H_k^n \longrightarrow \prod_{\substack{I_0 \in I \\ \sharp I_0 = n+1}} H^0\left(\bigcap_{i \in I_0} U_i, C_U \Phi(\mathcal{F})\right) \longrightarrow \varprojlim_k H_k^{n+1} \longrightarrow 0.$$

Hence it follows at once that for any $m \geq 1$ $\check{H}^m(\mathcal{U}, C_U \Phi(\mathcal{F})) = 0$. Lemma 6 is proved.

In the sequel we'll use the following variant of A. Kartan lemma, which connects the Čech cohomologies groups with the usual cohomologies groups of sheaves. For a sheaf \mathcal{A} on a topological space V by $\check{H}^q(V, \mathcal{A})$ denote direct limit of Čech cohomologies with respect to all coverings of the space V .

Lemma 7 *Let X be a topological space and \mathcal{A} be a sheaf on X . Suppose that it is possible to cover X by family \mathbf{U} of open sets such that this family has the following properties:*

1. *If \mathbf{U} contains U' and U'' , then it contains $U' \cap U''$;*
2. *\mathbf{U} contains arbitrarily small open sets;*
3. *$\check{H}^q(U, \mathcal{A}) = 0$ for any $q \geq 1$ and $U \in \mathbf{U}$.*

Under these conditions we have isomorphism:

$$\check{H}^q(X, \mathcal{A}) \longrightarrow H^q(X, \mathcal{A}).$$

Proof. See theorem 5.9.2 of [8, ch.2].

Lemma 8 *Let X be a noetherian scheme, $j : U \hookrightarrow X$ be an open affine subscheme. Then for any quasicoherent sheaf \mathcal{F} on U :*

$$H^0(X, j_* \mathcal{F}) = H^0(U, \mathcal{F}) \tag{18}$$

$$H^i(X, j_* \mathcal{F}) = 0 \quad \text{if } i > 0. \tag{19}$$

Proof.

Equality (18) follows from the construction of the functor j_* . Let us prove (19). Embed the quasicoherent sheaf \mathcal{F} in a flasque quasicoherent sheaf \mathcal{G} on U . (It always can do, see [9, ch. III, §3].)

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0$$

Now the following sequence is exact:

$$0 \longrightarrow j_* \mathcal{F} \longrightarrow j_* \mathcal{G} \longrightarrow j_*(\mathcal{G}/\mathcal{F}) \longrightarrow 0. \tag{20}$$

(Indeed, $R^1 j_* \mathcal{F} = 0$. The last follows from quasicoherentness of the sheaf $R^1 j_* \mathcal{F}$ and for any affine open $V \subset X$: $H^0(V, R^1 j_* \mathcal{F}) = H^1(V \cap U, \mathcal{F}) = 0$, as from separateness of X

it follows that $V \cap U$ is an affine scheme.) Besides, it is not difficult to see that the sheaf $j_*\mathcal{G}$ is an flasque sheaf. Therefore

$$H^i(X, j_*\mathcal{G}) = 0 \quad \text{for any } i > 0. \quad (21)$$

Besides, the map

$$H^0(X, j_*\mathcal{G}) \longrightarrow H^0(X, j_*(\mathcal{G}/\mathcal{F}))$$

is surjective, as $H^0(X, j_*\mathcal{G}) = H^0(U, \mathcal{G})$, $H^0(X, j_*(\mathcal{G}/\mathcal{F})) = H^0(U, \mathcal{G}/\mathcal{F})$. And from affineness of U it follows that $H^1(U, \mathcal{F}) = 0$, therefore the following map is surjective:

$$H^0(U, \mathcal{G}) \longrightarrow H^0(U, \mathcal{G}/\mathcal{F}).$$

Hence and from (21) we obtain that

$$H^1(X, j_*\mathcal{F}) = 0.$$

Further, if $i > 1$, then from (21) and from the long cohomological sequence associated with (20) it follows that

$$H^i(X, j_*\mathcal{F}) = H^{i-1}(X, j_*(\mathcal{G}/\mathcal{F})).$$

But the sheaf \mathcal{G}/\mathcal{F} is a quasicoherent sheaf on U . Hence, by induction, it is possible to assume that

$$H^{i-1}(X, j_*(\mathcal{G}/\mathcal{F})) = 0.$$

Therefore $H^i(X, j_*\mathcal{F}) = 0$. Lemma 8 is proved.

Lemma 9 *Let X be a noetherian scheme. Let $Y \hookrightarrow X$ be a closed subscheme, which is defined by the ideal sheaf J , and $j : U \hookrightarrow X$ be the open subscheme which is complement to the subscheme Y . Let a sheaf \mathcal{F} be a quasicoherent sheaf on X , and consider the following exact sequence of quasicoherent sheaves on X :*

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow j_*j^*\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

*which is induced by the natural map $\mathcal{F} \longrightarrow j_*j^*\mathcal{F}$.*

Let $\mathcal{H} = \varinjlim_i \mathcal{H}_i$ and $\mathcal{G} = \varinjlim_i \mathcal{G}_i$, $i \in I$, where the sheaves \mathcal{H}_i , \mathcal{G}_i are coherent sheaves on the scheme X for any $i \in I$.

Then for any $i \in I$ there exists $l(i) \in \mathbf{N}$ such that

$$J^{l(i)} \cdot \mathcal{H}_i = 0 \quad \text{and} \quad J^{l(i)} \cdot \mathcal{G}_i = 0. \quad (22)$$

Proof. From the exact sequence

$$0 \longrightarrow J^m \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/J^m \longrightarrow 0$$

it follows the following sequence of quasicoherent sheaves on X

$$0 \longrightarrow \mathcal{H}om_X(\mathcal{O}_X/J^m, \mathcal{F}) \longrightarrow \mathcal{H}om_X(\mathcal{O}_X, \mathcal{F}) \longrightarrow \mathcal{H}om_X(J^m, \mathcal{F}) \longrightarrow \mathcal{E}xt_X^1(\mathcal{O}_X/J^m, \mathcal{F}).$$

Taking direct limit with respect to m , we obtain

$$0 \longrightarrow \varinjlim_m \mathcal{H}om_X(\mathcal{O}_X/J^m, \mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \varinjlim_m \mathcal{H}om_X(J^m, \mathcal{F}) \longrightarrow \varinjlim_m \mathcal{E}xt_X^1(\mathcal{O}_X/J^m, \mathcal{F}).$$

By [9, ch. III, ex. 3.7(a)] we have

$$\varinjlim_m \mathcal{H}om_X(J^m, \mathcal{F}) = j_* j^* \mathcal{F}.$$

Now (22) follows from

$$\mathcal{H} = \varinjlim_m \mathcal{H}om_X(\mathcal{O}_X/J^m, \mathcal{F}) \quad \text{and} \quad \mathcal{G} \hookrightarrow \varinjlim_m \mathcal{E}xt_X^1(\mathcal{O}_X/J^m, \mathcal{F}).$$

Lemma 9 is proved.

3 Construction and its original properties

Let X be a noetherian separated scheme. Consider a flag of closed subschemes

$$X \supset Y_0 \supset Y_1 \supset \dots \supset Y_n$$

in the scheme X . Let J_j be the ideal sheaf of the subscheme Y_j in X ($0 \leq j \leq n$). Let i_j be the embedding of the subscheme $Y_j \hookrightarrow X$. Let U_i be an open subscheme of Y_i which is complement to the closed subscheme Y_{i+1} ($0 \leq i \leq n-1$). Let $j_i : U_i \hookrightarrow Y_i$ be the open embedding of the subscheme U_i to the scheme Y_i ($0 \leq i \leq n-1$). By definition, let $U_n = Y_n$ and j_n be the identity morphism from U_n to Y_n .

Assume that for any point $x \in X$ there exists an open affine neighbourhood $U \ni x$ such that $U \cap U_i$ is an affine scheme for any $0 \leq i \leq n$. In the sequel we'll say that a flag of subschemes $\{Y_i, 0 \leq i \leq n\}$ with such condition is the flag with *locally affine complements*.

Remark 3 For example, the last condition of locally affineness of complements is satisfied in the following cases

- Y_{i+1} is the Cartier divisor on the scheme Y_i ($0 \leq i \leq n-1$), or
- U_i is an affine scheme for any $0 \leq i \leq n-1$. (On a separated scheme the intersection of two open affine subschemes is an affine subscheme.)

Consider the n -dimensional simplex and its standard simplicial set (without degenerations). To be precise, consider the set:

$$(\{0\}, \{1\}, \dots, \{n\}).$$

(Here are all the integers between 0 and n .)

Then the simplicial set $S = \{S_k\}$:

- $S_0 \stackrel{\text{def}}{=} \{\eta \in \{0\}, \{1\}, \dots, \{n\}\}.$
- $S_k \stackrel{\text{def}}{=} \{(\eta_0, \dots, \eta_k), \text{ where } \eta_l \in S_0 \text{ and } \eta_{l-1} < \eta_l\}.$

The boundary map ∂_i ($0 < i < k$) is given by eliminating the i -th component of the vector (η_0, \dots, η_k) . (It is the i -th face of (η_0, \dots, η_k) .)

Definition. For any $(\eta_0, \dots, \eta_k) \in S_k$ define the functor

$$V_{(\eta_0, \dots, \eta_k)} : QS(X) \longrightarrow Sh(X)$$

uniquely determined by the following inductive conditions:

1. $V_{(\eta_0, \dots, \eta_k)}$ commutes with direct limits.
2. If \mathcal{F} is a coherent sheaf, and $\eta \in S_0$, then

$$V_{\eta}(\mathcal{F}) \stackrel{\text{def}}{=} \varprojlim_{m \in \mathbf{N}} (i_{\eta})_*(j_{\eta})^*(j_{\eta})^*(\mathcal{F}/J_{\eta}^m \mathcal{F}).$$

3. If \mathcal{F} is a coherent sheaf, and $(\eta_0, \dots, \eta_k) \in S_k$ ($k \geq 1$), then

$$V_{(\eta_0, \eta_1, \dots, \eta_k)}(\mathcal{F}) \stackrel{\text{def}}{=} \varprojlim_{m \in \mathbf{N}} V_{(\eta_1, \dots, \eta_k)} \left((i_{\eta_0})_*(j_{\eta_0})^*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \right).$$

In the sequel, to avoid the confusion of notations in the case of a lot of schemes and flags of closed subschemes we'll use sometimes the equivalent notation for $V_{(\eta_0, \dots, \eta_k)}(\mathcal{F})$, in which the closed subschemes is written explicitly:

$$V_{(\eta_0, \dots, \eta_k)}(\mathcal{F}) = V_{(Y_{\eta_0}, \dots, Y_{\eta_k})}(X, \mathcal{F}).$$

Proposition 1 Let $\sigma = (\eta_0, \dots, \eta_k) \in S_k$. Then

1. The functor $V_{\sigma} : QS(X) \longrightarrow Sh(X)$ is well defined.
2. The functor V_{σ} is exact and additive.

3. The functor V_σ is local on X , i. e., for any open $U \subset X$ for any quasicoherent sheaf \mathcal{F} on X :

$$V_{(Y_{\eta_0}, \dots, Y_{\eta_k})}(X, \mathcal{F})|_U = V_{(Y_{\eta_0} \cap U, \dots, Y_{\eta_k} \cap U)}(U, \mathcal{F}|_U).$$

(Here if $Y_j \cap U = \emptyset$, then $Y_j \cap U$ is an empty subscheme of U which is defined by the ideal sheaf \mathcal{O}_U .)

4. For any quasicoherent sheaf \mathcal{F} on the scheme X the sheaf $V_{(\eta_0, \dots, \eta_k)}(\mathcal{F})$ is a sheaf of \mathcal{O}_X -modules with the support on the subscheme Y_{η_k} . (Usually, this sheaf is not quasicoherent.)
5. For any quasicoherent sheaf \mathcal{F} on X :

$$V_\sigma(\mathcal{F}) = V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

6. If all U_i is affine ($0 \leq i \leq n$), then for any affine covering \mathcal{U} of the scheme X , for any quasicoherent sheaf \mathcal{F} on X , for any $m \geq 1$:

$$\check{H}^m(\mathcal{U}, V_\sigma(\mathcal{F})) = 0.$$

7. If all U_i is affine ($0 \leq i \leq n$), then for any quasicoherent sheaf \mathcal{F} on X , for any $m \geq 1$

$$H^m(X, V_\sigma(\mathcal{F})) = 0.$$

Proof.

1. Well-posedness of the definition of V_σ is proved by induction by means of using of lemma 1, lemma 2, lemma 3, lemma 5, lemma 6 and lemma 7. Let us check the base of induction for lemmas 5 and 6 (when the functor $\Phi = id$). Namely

- a) for any affine scheme \mathcal{V} and any quasicoherent sheaf \mathcal{F} on V

$$H^1(V, \mathcal{F}) = 0.$$

- b) Let us show that if $i : Y \hookrightarrow X$ is a closed subscheme with the ideal sheaf J , $j : U \hookrightarrow Y$ is an open imbedding of the affine scheme U in Y . Then for any quasicoherent sheaf \mathcal{F} on X , for any $k \geq 1$, for any affine open covering \mathcal{U} of the scheme X , for any $m \geq 1$:

$$\check{H}^m(\mathcal{U}, i_* j_* j^*(\mathcal{F}/J^k \mathcal{F})) = 0.$$

From affineness of the covering \mathcal{U} it follows that it is acyclic for quasicoherent sheaves. Consequently the Čech cohomologies groups with respect to this covering coincide with the usual cohomologies groups of quasicoherent sheaves. (See [9, ch.3, theorem 4.5]) Therefore it suffices to prove that for any integer $k \geq 1$

$$H^m(X, i_* j_* j^*(\mathcal{F}/J^k \mathcal{F})) = 0. \quad (23)$$

If $k = 1$, then the sheaf $\mathcal{F}/J\mathcal{F}$ is quasicoherent with respect to the subscheme Y , and

$$H^m(X, i_* j_* j^*(\mathcal{F}/J\mathcal{F})) = H^m(Y, j_*(j^*(\mathcal{F}/J\mathcal{F}))) = 0.$$

Where the last equality follows from lemma 8.

If $k > 1$, then by lemma 4 the following sequence is exact

$$0 \longrightarrow i_* j_* j^*(J^{k-1}\mathcal{F}/J^k\mathcal{F}) \longrightarrow i_* j_* j^*(\mathcal{F}/J^k\mathcal{F}) \longrightarrow i_* j_* j^*(\mathcal{F}/J^{k-1}\mathcal{F}) \longrightarrow 0. \quad (24)$$

In addition, the sheaves $J^{k-1}\mathcal{F}/J^k\mathcal{F}$ and $\mathcal{F}/J^{k-1}\mathcal{F}$ are quasicoherent with respect to the subscheme $(Y, \mathcal{O}_X/J^{k-1})$. Therefore, by induction, we obtain

$$H^m(X, i_* j_* j^*(J^{k-1}\mathcal{F}/J^k\mathcal{F})) = 0 \quad \text{and} \quad H^m(X, i_* j_* j^*(\mathcal{F}/J^{k-1}\mathcal{F})) = 0.$$

Hence, from (24) we have (23). Item 1 of proposition 1 is proved.

2. The proof of this item is analogous to the proof of item 1 by means of the same lemmas.
3. Localness follows by induction from the construction of the functor $V_{(\eta_0, \dots, \eta_k)}$.
4. This item follows by induction from the construction.
5. We have the natural map:

$$\mathcal{F} \longrightarrow V_\sigma(\mathcal{F}),$$

which induces the following map:

$$V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} V_\sigma(\mathcal{F}) \longrightarrow V_\sigma(\mathcal{O}_X) \otimes_{V_\sigma(\mathcal{O}_X)} V_\sigma(\mathcal{F}) = V_\sigma(\mathcal{F}). \quad (25)$$

Let us show that (25) gives us an isomorphism between $V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F}$ and $V_\sigma(\mathcal{F})$. Since the functor V_σ and tensor products commute with direct limits, we can assume that \mathcal{F} is a coherent sheaf. In view of item 3 of this proposition, we can restrict ourself to the local situation. That is, we suppose $X = \text{Spec } A$, $\mathcal{F} = \tilde{M}$ for some finitely generated A -module M . Then for some $r \in \mathbf{N}$ there exists an exact sequence of sheaves as:

$$0 \longrightarrow \tilde{N} \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \tilde{M} \longrightarrow 0,$$

where N is some finitely generated A -module. Hence we obtain the commutative diagram:

$$\begin{array}{ccccccc} V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \tilde{N} & \longrightarrow & V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus r} & \longrightarrow & V_\sigma(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \tilde{M} & \longrightarrow & 0 \\ \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha & & \\ 0 \longrightarrow & V_\sigma(\tilde{N}) & \longrightarrow & V_\sigma(\mathcal{O}_X^{\oplus r}) & \xrightarrow{\delta} & V_\sigma(\tilde{M}) & \longrightarrow 0, \end{array}$$

where the lower row is exact by virtue of item 2. Besides, it is clear that β is an isomorphism. Therefore from surjectivity of δ it follows that α is surjective. Since \tilde{N} is a coherent sheaf, we have that the map γ is surjective as well. Hence, from exactness of

the lower row and non complicated diagram search it follows that the map α is injective.

6. The proof is similar to the proof of item 1 by means of the same lemmas.

7. This item follows from the previous item of this proposition and lemma 7. (Since every point has arbitrarily small affine neighbourhood with affine intersection to all U_i , we obtain that this affine neighbourhood satisfies item 6 of proposition 1.)

Proposition 2 1. Let X be a noetherian separated scheme. Let

$$Y_0 \supset Y_1 \supset \dots \supset Y_n \quad \text{and} \quad Y'_0 \supset Y'_1 \supset \dots \supset Y'_n$$

be two flags of closed subschemes in X with the corresponding ideal sheaves J_i and J'_i ($0 \leq i \leq n$) such that for any $0 \leq i \leq n$ there exist integers $l_i \geq 1$ and $l'_i \geq 1$ with the following properties:

$$J_i^{l_i} \subset J'_i \quad \text{and} \quad (J'_i)^{l'_i} \subset J_i. \quad (26)$$

Then the functors

$$V_{(Y_{\eta_0}, \dots, Y_{\eta_k})}(X, \cdot) \quad \text{and} \quad V_{(Y'_{\eta_0}, \dots, Y'_{\eta_k})}(X, \cdot)$$

coincides for any $(\eta_0, \dots, \eta_k) \in S_k$.

2. Consider the flag of closed subschemes:

$$X \supset Z \supset Y_0 \supset \dots \supset Y_n$$

on a noetherian separated scheme X . Let $i : Z \hookrightarrow X$ be a closed imbedding. Then for any quasicoherent sheaf \mathcal{F} on the scheme Z we have

$$i_* \left(V_{(Y_0, \dots, Y_n)}(Z, \mathcal{F}) \right) = V_{(Y_0, \dots, Y_n)}(X, i_* \mathcal{F}).$$

Remark 4 Condition (26) is equivalent to the statement that the topological spaces of subschemes Y_i and Y'_i are the same for all $0 \leq i \leq n$.

Proof (of proposition 2).

1. It suffices to prove for the case when $J_i = J'_i$ for all $i \neq j$, $0 \leq i \leq n$, where some fixing $0 \leq j \leq n$. From inductance of the definition of the functor $V_{(Y_{\eta_0}, \dots, Y_{\eta_k})}(X, \cdot)$ (and $V_{(Y'_{\eta_0}, \dots, Y'_{\eta_k})}(X, \cdot)$) we can restrict ourself to the case $j = 0$. This case follows from cofinality of the projective systems \mathcal{F}/J_j^k and $\mathcal{F}/(J'_j)^k$ (from condition 26).

2. This item follows at once from the construction and the fact that supports of all appearing from induction sheaves are on the subscheme Z .

4 Complexes and their exactness

Consider again the usual n -simplex without degenerations $S = \{S_k, 0 \leq k \leq n\}$. If $\sigma = (\eta_0, \dots, \eta_k) \in S_k$, then $\partial_i(\sigma)$ is the i -th face of σ ($0 \leq i \leq k$). Then *define* the morphism of functors

$$d_i(\sigma) : V_{\partial_i(\sigma)} \longrightarrow V_\sigma, \quad \text{as}$$

commuting with direct limits, and on coherent sheaves it is a map

$$V_{\partial_i(\sigma)}(\mathcal{F}) \longrightarrow V_\sigma(\mathcal{F}) \tag{27}$$

which is defined by the following rules:

- a) if $i = 0$, then (27) is obtained from application of the functor $V_{\partial_0(\sigma)}$ to the map

$$\mathcal{F} \longrightarrow (i_{\eta_0})_*(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})$$

and passage to the projective limit on m ;

- b) if $i = 1, k = 1$, then we have the natural map

$$(i_{\eta_0})_*(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \longrightarrow V_{(\eta_1)}((i_{\eta_0})_*(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})).$$

Now after passage to the projective limit on m we obtain the map (27) in this case.

- c) $i \neq 0, k > 1$, then from induction on k we can suppose that we have the map

$$V_{\partial_{i-1}(\partial_0(\sigma))}((i_{\eta_0})_*(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \longrightarrow V_{\partial_0(\sigma)}((i_{\eta_0})_*(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})).$$

And passage to the projective limit on m gives us the map (27) in this case.

Proposition 3 *For any $1 \leq k \leq n, 0 \leq i \leq k$ define*

$$d_i^k \stackrel{\text{def}}{=} \sum_{\sigma \in S_k} d_i(\sigma) : \bigoplus_{\sigma \in S_{k-1}} V_\sigma \longrightarrow \bigoplus_{\sigma \in S_k} V_\sigma.$$

Also define

$$d_0^0 : id \longrightarrow \bigoplus_{\sigma \in S_0} V_\sigma$$

as the direct sum of the natural maps $\mathcal{F} \longrightarrow V_\sigma(\mathcal{F})$. (Here id is the functor of the natural imbedding of $QS(X)$ into $Sh(X)$, \mathcal{F} is a quasicoherent sheaf on X , $\sigma \in S_0$.)

Then for all $0 \leq i < j \leq k \leq n-1$ we have

$$d_j^{k+1} d_i^k = d_i^{k+1} d_{j-1}^k. \tag{28}$$

Proof. Using the inductance of the definition, the proof is done by induction from non complicated consideration of some cases. It suffices to consider the small i and k only. (For example, see similar cases in [11, §2.4] or [10].)

Define

$$d^m \stackrel{\text{def}}{=} \sum_{0 \leq i \leq m} (-1)^i d_i^m$$

Then proposition 3 makes possible to construct the complex of sheaves $V(\mathcal{F})$ from any quasicoherent sheaf \mathcal{F} on X in the standard way:

$$\dots \longrightarrow \bigoplus_{\sigma \in S_{m-1}} V_{\sigma}(\mathcal{F}) \xrightarrow{d^m} \bigoplus_{\sigma \in S_m} V_{\sigma}(\mathcal{F}) \longrightarrow \dots$$

Where $d^{m+1}d^m = 0$ follows from (28) by means of non complicated direct calculations.

Theorem 1 *Let X be a noetherian separated scheme. Let $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes with locally affine complements. Assume that $Y_0 = X$. Then the following complex is exact:*

$$0 \longrightarrow \mathcal{F} \xrightarrow{d^0} V(\mathcal{F}) \longrightarrow 0. \quad (29)$$

Proof. It suffices to consider only the case when the sheaf \mathcal{F} is coherent. Consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{F} \longrightarrow (j_0)_*(j_0)^*\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \quad (30)$$

Here \mathcal{H} and \mathcal{G} is the kernel and the cokernel of the natural map of sheaves $\mathcal{F} \longrightarrow (j_0)_*(j_0)^*\mathcal{F}$. From exactness of functors V_{σ} (for any σ) we obtain the following exact sequence of complexes of sheaves:

$$0 \longrightarrow V(\mathcal{H}) \longrightarrow V(\mathcal{F}) \longrightarrow V((j_0)_*(j_0)^*\mathcal{F}) \longrightarrow V(\mathcal{G}) \longrightarrow 0 \quad (31)$$

By lemma 9 the supports of sheaves \mathcal{H} and \mathcal{G} are on Y_1 , therefore in the case $\eta_0 = 0$ we have $V_{(Y_{\eta_0}, \dots, Y_{\eta_k})}(X, \mathcal{H}) = 0$ and $V_{(Y_{\eta_0}, \dots, Y_{\eta_k})}(X, \mathcal{G}) = 0$. Therefore, using it, lemma 9 (which decompose the sheaves \mathcal{H} and \mathcal{G} in direct limits of sheaves which is coherent on subschemes with topological space Y_1), proposition 2, permutability of the functors V_{σ} with direct limits, we can apply induction on the length of flag and suppose that the complexes

$$0 \longrightarrow \mathcal{H} \xrightarrow{d^0} V(\mathcal{H}) \longrightarrow 0 \quad \text{and} \quad (32)$$

$$0 \longrightarrow \mathcal{G} \xrightarrow{d^0} V(\mathcal{G}) \longrightarrow 0 \quad (33)$$

are already exact. It is not difficult to understand that for any $\sigma = (\eta_0, \dots, \eta_k) \in S_k$ if $\eta_0 = 0$, then $V_{\sigma}((j_0)_*(j_0)^*\mathcal{F}) = V_{\sigma}(\mathcal{F})$; if $\eta_0 \neq 0$, then $V_{\sigma}((j_0)_*(j_0)^*\mathcal{F}) = V_{\sigma'}(\mathcal{F})$, where $\sigma' = (0, \eta_0, \dots, \eta_k) \in S_{k+1}$. Hence, the complex $V((j_0)_*(j_0)^*\mathcal{F})$ has the same components

$V_{\sigma'}(\mathcal{F})$ in the degree k and $k+1$. Therefore, successively from the highest degrees splitting off the trivial complexes

$$0 \longrightarrow V_{\sigma}((j_0)_*(j_0)^*\mathcal{F}) \longrightarrow V_{\sigma'}((j_0)_*(j_0)^*\mathcal{F}) \longrightarrow 0,$$

we obtain exactness of the complex

$$0 \longrightarrow (j_0)_*(j_0)^*\mathcal{F} \xrightarrow{d^0} V((j_0)_*(j_0)^*\mathcal{F}) \longrightarrow 0. \quad (34)$$

Now, since complexes (32), (33), (34) are exact, we obtain exactness of complex (29) from exactness of (31) and (30). Theorem 1 is proved.

For any $\sigma \in S_k$ define

$$A_{\sigma}(\mathcal{F}) \stackrel{\text{def}}{=} H^0(X, V_{\sigma}(\mathcal{F})).$$

Proposition 4 *Let X be a noetherian separated scheme. $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes such that all U_i are affine ($0 \leq i \leq n$). Let $\sigma \in S_k$ be arbitrary. Then*

1. A_{σ} is an exact and additive functor: $QS(X) \longrightarrow Ab$.
2. If $X = \text{Spec } A$, M is some A -module, then

$$A_{\sigma}(\tilde{M}) = A_{\sigma}(\mathcal{O}_X) \otimes_A M.$$

Proof.

1. This item follows at once from items 2 and 7 of proposition 1.
2. Similarly to the proof of item 5 of proposition 1 we can suppose that the module M is finitely generated over A . Now consider the exact sequence of A -modules:

$$0 \longrightarrow N \longrightarrow A^{\oplus r} \longrightarrow M \longrightarrow 0.$$

Hence we obtain the commutative diagramm:

$$\begin{array}{ccccccc} A_{\sigma}(\mathcal{O}_X) \otimes_A \tilde{N} & \longrightarrow & A_{\sigma}(\mathcal{O}_X)^{\oplus r} & \longrightarrow & A_{\sigma}(\mathcal{O}_X) \otimes_A M & \longrightarrow & 0 \\ \downarrow \gamma & & \downarrow \beta & & \downarrow \alpha & & \\ 0 \longrightarrow & A_{\sigma}(\tilde{N}) & \longrightarrow & A_{\sigma}(\mathcal{O}_X^{\oplus r}) & \xrightarrow{\delta} & A_{\sigma}(\tilde{M}) & \longrightarrow 0, \end{array}$$

where the lower row is exact by virtue of item 1 of this proposition. It is clear that β is an isomorphism. Therefore, arguing as in item 5, we obtain at first surjectivity of the map α , and afterwards we obtain injectivity of α . Proposition 4 is proved.

Let \mathcal{F} be any quasicoherent sheaf on X . Apply the functor $H^0(X, \cdot)$ to the complex $V(\mathcal{F})$. We obtain the complex of abelian groups $A(\mathcal{F})$:

$$\dots \longrightarrow \bigoplus_{\sigma \in S_{m-1}} A_{\sigma}(\mathcal{F}) \longrightarrow \bigoplus_{\sigma \in S_m} A_{\sigma}(\mathcal{F}) \longrightarrow \dots$$

Theorem 2 *Let X be a noetherian separated scheme. Let $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes such that $Y_0 = X$ and all U_i are affine ($0 \leq i \leq n$). Then cohomology of the complex $A(\mathcal{F})$ coincide with cohomology of the sheaf \mathcal{F} on X , i. e., for any i*

$$H^i(X, \mathcal{F}) = H^i(A(\mathcal{F})).$$

Proof. From theorem 1 and item 7 of proposition 1 it follows that $V(\mathcal{F})$ is an acyclic resolution for the sheaf \mathcal{F} . Therefore it is possible to calculate cohomology of the sheaf \mathcal{F} by means of global sections of this resolution. Theorem 2 is proved.

From the last theorem we obtain at once the following geometrical corollary.

Theorem 3 *Let X be a projective algebraic scheme of dimension n over a field. Let $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes such that $Y_0 = X$ and Y_i is an ample divisor on the scheme Y_{i-1} for any $1 \leq i \leq n$. Then for any quasicoherent sheaf \mathcal{F} on X , for any i we have*

$$H^i(X, \mathcal{F}) = H^i(A(\mathcal{F})).$$

Proof. In fact, since Y_i is an ample divisor on Y_{i-1} for all $1 \leq i \leq n$, we have that U_i is an affine scheme for all $0 \leq i \leq n-1$. Since $\dim Y_n = 0$, we have that $U_n = Y_n$ is affine as well. Now application of theorem 2 concludes the proof.

Remark 5 Let us remark that for any quasicoherent sheaf \mathcal{F} , for any $\sigma = (\eta_0) \in S_0$ $A_{\sigma}(\mathcal{F})$ is the group of section over U_{η_0} of the sheaf \mathcal{F} lifted to the formal neighbourhood of the subscheme Y_{η_0} in X . And the complex $A(\mathcal{F})$ can be interpreted as the Čech complex for the such "covering" of the scheme X .

5 Combinatorial properties and the Krichever map.

Lemma 10 *Let X be a noetherian separated scheme. Let $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes such that $Y_0 = X$ and Y_i is an ample Cartier divisor on the scheme Y_{i-1} ($1 \leq i \leq n$). Let J_i be the ideal sheaves on X defining the corresponding subschemes Y_i in X . Let $\sigma = (\eta_0, \dots, \eta_k) \in S_k$. Then for any $i \leq \eta_0$, for any quasicoherent sheaf \mathcal{F} on X we have*

$$A_{\sigma}(\mathcal{F}) = \varprojlim_m A_{\sigma}(\mathcal{F}/J_i^m \mathcal{F}). \quad (35)$$

Remark 6 We consider the sheaf $\mathcal{F}/J_i^m \mathcal{F}$ in (35) as the sheaf on the scheme X . The corresponding functor of direct image from the subscheme Y_i is omitted for the sake of simplification of notations. Further we shall do the same in analogous situations.

Proof. From the definition of the functor A_σ we have

$$\begin{aligned} \varprojlim_m A_\sigma(\mathcal{F}/J_i^m \mathcal{F}) &= \varprojlim_m \varprojlim_l A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/(J_i^m + J_{\eta_0}^l) \mathcal{F})) = \\ &= \varprojlim_{(m,l)} A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/(J_i^m + J_{\eta_0}^l) \mathcal{F})) = \varprojlim_l A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^l \mathcal{F})) = A_\sigma(\mathcal{F}), \end{aligned}$$

where next to the last equality follows from cofinality of the projective systems

$$A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/(J_i^m + J_{\eta_0}^l) \mathcal{F})) \quad \text{and} \quad A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^l \mathcal{F})).$$

The last follows from cofinality of the systems $\mathcal{F}/(J_i^m + J_{\eta_0}^l) \mathcal{F}$ and $\mathcal{F}/J_{\eta_0}^l \mathcal{F}$. Besides, in our reasonings we meant that if $\sigma \in S_0$, then $A_{\partial_0(\sigma)} = H^0(X, \cdot)$. The lemma is proved.

Lemma 11 *Let X be a Cohen-Macaulay noetherian scheme. Let $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes such that $Y_0 = X$ and Y_i is a Cartier divisor on the scheme Y_{i-1} ($1 \leq i \leq n$). Let J_i be the ideal sheaves on X defining the corresponding subschemes Y_i in X . Let j_k be an open imbedding of $Y_k \setminus Y_{k+1}$ into Y_k ($0 \leq k < n$). Then for any $0 \leq k < n$, for any $m \geq 1$, for any locally free sheaf \mathcal{F} on X the natural map*

$$\mathcal{F}/J_k^m \mathcal{F} \longrightarrow (j_k)_*(j_k)^*(\mathcal{F}/J_k^m \mathcal{F})$$

is an imbedding.

Proof. Let us do induction on m .

Let $m = 1$. Then $\mathcal{F}/J_k \mathcal{F}$ is a locally free sheaf on Y_k . Therefore, applying lemma 9 to the pair $Y_{k+1} \hookrightarrow Y_k$, we obtain injectivity in this case, since Cartier divisor is locally generated by one element which is not divisor of zero in the structure sheaf.

Let $m > 1$. Then since Y_i are Cartier divisors on Y_{i-1} , we have that Y_k is a local complete intersection in X and locally defined by a regular sequence on X . Therefore the sheaf J_k^{m-1}/J_k^m is locally free on Y_k (see [9, ch. II, th. 8.21A]). From the exact sequence

$$0 \longrightarrow J_k^{m-1} \mathcal{F}/J_k^m \mathcal{F} \longrightarrow \mathcal{F}/J_k^m \mathcal{F} \longrightarrow \mathcal{F}/J_k^{m-1} \mathcal{F} \longrightarrow 0 \quad (36)$$

and lemma 4 we obtain exactness of the following sequence

$$0 \rightarrow (j_k)_*(j_k)^*(J_k^{m-1} \mathcal{F}/J_k^m \mathcal{F}) \rightarrow (j_k)_*(j_k)^*(\mathcal{F}/J_k^m \mathcal{F}) \rightarrow (j_k)_*(j_k)^*(\mathcal{F}/J_k^{m-1} \mathcal{F}) \rightarrow 0. \quad (37)$$

The sheaf $J_k^{m-1} \mathcal{F}/J_k^m \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} J_k^{m-1}/J_k^m$ is locally free Y_k . Therefore the map

$$J_k^{m-1} \mathcal{F}/J_k^m \mathcal{F} \longrightarrow (j_k)_*(j_k)^*(J_k^{m-1} \mathcal{F}/J_k^m \mathcal{F})$$

is an imbedding. Also the map

$$\mathcal{F}/J_k^{m-1} \mathcal{F} \longrightarrow (j_k)_*(j_k)^*(\mathcal{F}/J_k^{m-1} \mathcal{F})$$

is an imbedding by induction hypothesis. From this, (36), (37) and non complicated diagram search we obtain that the map

$$\mathcal{F}/J_k^m \mathcal{F} \longrightarrow (j_k)_*(j_k)^*(\mathcal{F}/J_k^m \mathcal{F})$$

is an imbedding. The lemma is proved.

Lemma 12 *Let X be a projective equidimensional Cohen-Macaulay algebraic scheme of dimension n over a field. Let $Y_0 \supset Y_1 \supset \dots \supset Y_k$ be a flag of closed subschemes such that $Y_0 = X$ and Y_i is an ample Cartier divisor on the scheme Y_{i-1} for any i ($1 \leq i \leq k$). Let J_i be the ideal sheaves on X defining the corresponding subschemes Y_i in X . Then for any locally free sheaf \mathcal{F} on X the natural map*

$$H^0(X, \mathcal{F}) \longrightarrow \varprojlim_m H^0(X, \mathcal{F}/J_k^m \mathcal{F})$$

is an imbedding; and if $k < n$, then this one is an isomorphism.

Proof. The proof will be done by induction on k .

Let at first $k = 1$. The sheaf J_1 is the dual of the ample invertible sheaf on X . And from conditions on X (that is Cohen-Macaulayness, projectiveness, equidimensionality) there exist (see [9, ch. III, th. 7.6]) $l > 0$ such that for any $m > l$ we have $H^0(X, J_1^m \mathcal{F}) = 0$, and if $n \geq 2$, then we have $H^1(X, J_1^m \mathcal{F}) = 0$ as well.

Hence and from the exact sequence

$$0 \longrightarrow J_1^m \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/J_1^m \mathcal{F} \longrightarrow 0$$

we obtain that the map

$$H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F}/J_1^m \mathcal{F})$$

is an imbedding for $m > l$, and

$$H^0(X, \mathcal{F}) = H^0(X, \mathcal{F}/J_1^m \mathcal{F})$$

for $m > l$ and $n \geq 2$. And after passage to the projective limit on m we obtain the imbedding

$$H^0(X, \mathcal{F}) \hookrightarrow \varprojlim_m H^0(X, \mathcal{F}/J_1^m \mathcal{F}) = H^0(X, \varprojlim_m \mathcal{F}/J_1^m \mathcal{F}),$$

and if $n > 1$, then this one is the isomorphism

$$H^0(X, \mathcal{F}) \simeq H^0(X, \varprojlim_m \mathcal{F}/J_1^m \mathcal{F}).$$

Now let $k > 1$ be arbitrary. From lemmas conditions it follows that Y_k is locally defined by a regular sequence on X . Therefore Y_k is a Cohen-Macaulay scheme, and J_k^m/J_k^{m+1} are locally free sheaves on Y_k (see [9, ch. II, th. 8.21A]). By induction hypothesis we have

$$\varprojlim_l H^0(X, \mathcal{F}/J_{k-1}^l \mathcal{F}) = H^0(X, \mathcal{F}). \quad (38)$$

From cofinality of projective systems $\mathcal{F}/J_k^m \mathcal{F}$ and $\mathcal{F}/(J_k^m + J_{k-1}^l) \mathcal{F}$ we have that

$$\lim_{\leftarrow m} H^0(X, \mathcal{F}/J_k^m \mathcal{F}) = \lim_{(l,m)} H^0(X, \mathcal{F}/(J_k^m + J_{k-1}^l) \mathcal{F}) = \lim_l \lim_m H^0(X, \mathcal{F}/(J_k^m + J_{k-1}^l) \mathcal{F}).$$

From this one and equality (38) it suffices for the proof of the lemma to show that for any $l \geq 1$ the map

$$H^0(X, \mathcal{F}/J_{k-1}^l \mathcal{F}) \longrightarrow \lim_m H^0(X, \mathcal{F}/(J_k^m + J_{k-1}^l) \mathcal{F})$$

is an imbedding, and if $k < n$ then this map is an isomorphism.

For this one let us consider the exact sequence

$$0 \longrightarrow \frac{J_k^m \mathcal{F} + J_{k-1}^l \mathcal{F}}{J_{k-1}^l \mathcal{F}} \longrightarrow \mathcal{F}/J_{k-1}^l \mathcal{F} \longrightarrow \mathcal{F}/(J_k^m + J_{k-1}^l) \mathcal{F} \longrightarrow 0.$$

It suffices to show that for all sufficiently large m

$$H^0(X, \frac{J_k^m \mathcal{F} + J_{k-1}^l \mathcal{F}}{J_{k-1}^l \mathcal{F}}) = 0, \quad (39)$$

and if $k < n$, then

$$H^1(X, \frac{J_k^m \mathcal{F} + J_{k-1}^l \mathcal{F}}{J_{k-1}^l \mathcal{F}}) = 0. \quad (40)$$

For this one let us do induction on l . If $l = 1$, then (39) and (40) follows at once from [9, ch. III, th. 7.6] and the fact that the sheaf J_k/J_{k-1} is the dual of the ample invertible sheaf on Y_{k-1} .

If $l > 1$, then from the identity $\frac{A+B}{B} = \frac{A}{A \cap B}$ it follows the exact sequence

$$0 \longrightarrow \frac{J_k^m \cap J_{k-1}^{l-1}}{J_k^m \cap J_{k-1}^l} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \frac{J_k^m \mathcal{F} + J_{k-1}^l \mathcal{F}}{J_{k-1}^l \mathcal{F}} \longrightarrow \frac{J_k^m \mathcal{F} + J_{k-1}^{l-1} \mathcal{F}}{J_{k-1}^{l-1} \mathcal{F}} \longrightarrow 0.$$

Restricting ourself to the local situation and using the fact that J_i is generated by a regular sequence and [9, ch. II, th. 8.21A], it is not difficult to understand that

$$\frac{J_k^m \cap J_{k-1}^{l-1}}{J_k^m \cap J_{k-1}^l} \otimes_{\mathcal{O}_X} \mathcal{F} = \frac{J_k^{m-l+1} + J_{k-1}}{J_{k-1}} \cdot \left(\frac{J_{k-1}^{l-1}}{J_{k-1}^l} \otimes_{\mathcal{O}_X} \mathcal{F} \right).$$

In addition, the sheaf $\frac{J_{k-1}^{l-1}}{J_{k-1}^l} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free on Y_{k-1} . The sheaf J_k/J_{k-1} is the dual of the ample invertible sheaf on the Cohen-Macaulay scheme Y_{k-1} . Therefore from [9, ch. III, th. 7.6] we have for sufficiently large m that:

$$H^0(X, \frac{J_k^m \cap J_{k-1}^{l-1}}{J_k^m \cap J_{k-1}^l} \otimes_{\mathcal{O}_X} \mathcal{F}) = 0 \quad , \text{ and}$$

$$\text{if } k < n \quad , \text{ then } \quad H^1(X, \frac{J_k^m \cap J_{k-1}^{l-1}}{J_k^m \cap J_{k-1}^l} \otimes_{\mathcal{O}_X} \mathcal{F}) = 0.$$

The lemma is proved.

Corollary(from lemma 12)

Under the conditions of lemma 12 for any $\sigma \in S_0$ the natural map

$$H^0(X, \mathcal{F}) \longrightarrow A_\sigma(\mathcal{F})$$

is an imbedding.

Proof. Let $\sigma = (m)$. By lemma 12 we have the imbedding

$$0 \longrightarrow \mathcal{F}/J_k^m \mathcal{F} \longrightarrow (j_k)_*(j_k)^*(\mathcal{F}/J_k^m \mathcal{F}).$$

Hence we obtain the imbedding

$$0 \longrightarrow H^0(X, \mathcal{F}/J_k^m \mathcal{F}) \longrightarrow H^0(X, (j_k)_*(j_k)^*(\mathcal{F}/J_k^m \mathcal{F})).$$

After passage to the projective limit on m we obtain the imbedding

$$0 \longrightarrow \varprojlim_m H^0(X, \mathcal{F}/J_k^m \mathcal{F}) \longrightarrow A_\sigma(\mathcal{F}).$$

Now application of lemma 12 concludes the proof of the corollary.

Theorem 4 *Let X be a projective equidimensional Cohen-Macaulay scheme of dimension n over a field. Let $Y_0 \supset Y_1 \supset \dots \supset Y_n$ be a flag of closed subschemes such that $Y_0 = X$ and Y_i is an ample Cartier divisor on the scheme Y_{i-1} for any $1 \leq i \leq n$. Then for any locally free sheaf \mathcal{F} on X we have that*

1. *for any $\sigma \in S_k$ ($0 \leq k \leq n$) the natural map*

$$H^0(X, \mathcal{F}) \longrightarrow A_\sigma(\mathcal{F})$$

is an imbedding,

2. *for any $\sigma \in S_k$ ($1 \leq k \leq n$), for any i ($0 \leq i \leq k$) the natural map*

$$d_i(\sigma) : A_{\partial_i(\sigma)}(\mathcal{F}) \longrightarrow A_\sigma(\mathcal{F})$$

is an imbedding.

Remark 7 Taking into account (28) from proposition 3, it is possible to reformulate item 2 of this theorem in the following way:

for any locally free sheaf \mathcal{F} on X , for any $\sigma_1, \sigma_2 \in S$, $\sigma_1 \subset \sigma_2$ the natural map $A_{\sigma_1}(\mathcal{F}) \longrightarrow A_{\sigma_2}(\mathcal{F})$ is an imbedding.

Proof. Let J_i be the ideal sheaves on X defining the corresponding subschemes Y_i in X . Let us prove first item 2 of the theorem. Consider 3 cases.

Case 1. Let $\sigma = (\eta_0, \eta_1, \dots, \eta_k)$, and $i = 0$. Then $\partial_0(\sigma) = (\eta_1, \dots, \eta_k)$.

By lemma 11 for any $m \geq 1$ the map

$$\mathcal{F}/J_{\eta_0}^m \mathcal{F} \longrightarrow (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})$$

is an imbedding. Apply to this sequence the exact functor $A_{\partial_0(\sigma)}$. We obtain the imbedding:

$$A_{\partial_0(\sigma)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})).$$

After passage to the projective limit on m we obtain the imbedding

$$\varprojlim_m A_{(\eta_1, \dots, \eta_k)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \longrightarrow A_{\sigma}(\mathcal{F}).$$

In addition, from lemma 10 we have that

$$\varprojlim_m A_{(\eta_1, \dots, \eta_k)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) = A_{(\eta_1, \dots, \eta_k)}(\mathcal{F}).$$

Thus we obtain that in this case the map

$$A_{\partial_0(\sigma)}(\mathcal{F}) \longrightarrow A_{\sigma}(\mathcal{F})$$

is an imbedding.

Case 2. Let $\sigma = (\eta_0, \eta_1)$, and $i = 1$. In this case we have that

$$A_{\partial_1(\sigma)}(\mathcal{F}) = \varprojlim_m H^0(X, (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F}))$$

$$A_{\sigma}(\mathcal{F}) = \varprojlim_m A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})),$$

and for the proof of this case it suffices to show that for any $m \geq 1$ the map

$$H^0(X, (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \quad (41)$$

is an imbedding.

Let us show this by induction. Let $m = 1$. Then $\mathcal{F}/J_{\eta_0} \mathcal{F}$ is a locally free sheaf on Y_{η_0} . Besides, since Y_{η_0+1} is a Cartier divisor on Y_{η_0} , we have from item 5 of proposition 1 that

$$\begin{aligned} (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0} \mathcal{F}) &= ((j_{\eta_0})_*(j_{\eta_0})^* \mathcal{O}_{Y_{\eta_0}}) \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{F}/J_{\eta_0} \mathcal{F}) = \\ &= \varinjlim_j \mathcal{O}_{Y_{\eta_0}}(jY_{\eta_0+1}) \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{F}/J_{\eta_0} \mathcal{F}) = \varinjlim_j (\mathcal{F}/J_{\eta_0} \mathcal{F})(jY_{\eta_0+1}). \end{aligned} \quad (42)$$

The sheaves $(\mathcal{F}/J_{\eta_0} \mathcal{F})(jY_{\eta_0+1})$ are locally free on Y_{η_0} as well.

Therefore by corollary from lemma 12 the map

$$H^0(X, (\mathcal{F}/J_{\eta_0} \mathcal{F})(jY_{\eta_0+1})) \longrightarrow A_{\partial_0(\sigma)}((\mathcal{F}/J_{\eta_0} \mathcal{F})(jY_{\eta_0+1}))$$

is an imbedding. After passage to the projective limit on j we obtain injectivity of (41) in the case $m = 1$.

If $m > 1$, then the statement will follow from induction hypothesis and consideration of the following two exact sequences:

$$\begin{aligned} 0 \longrightarrow H^0(X, (j_{\eta_0})_*(j_{\eta_0})^*\mathcal{G}) &\longrightarrow H^0(X, (j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^m\mathcal{F}) \longrightarrow \\ &\longrightarrow H^0(X, (j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^{m-1}\mathcal{F}) \longrightarrow 0 \\ 0 \longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{G}) &\longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^m\mathcal{F}) \longrightarrow \\ &\longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^{m-1}\mathcal{F}) \longrightarrow 0, \end{aligned}$$

where the sheaf $\mathcal{G} = J_{\eta_0}^{m-1}\mathcal{F}/J_{\eta_0}^m\mathcal{F}$ is a locally free on Y_{η_0} . Case 2 is analyzed.

Case 3. We shall consider all that are not in cases 1 and 2. Let $\sigma = (\eta_0, \eta_1, \dots, \eta_k)$ and $i \neq 0$. Let us do induction on i . The case $i = 0$ is already analyzed (case 1). We have

$$\begin{aligned} A_{\partial_i(\sigma)}(\mathcal{F}) &= \lim_{\longleftarrow m} A_{\partial_{i-1}(\partial_0(\sigma))}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m\mathcal{F})) \quad \text{and} \\ A_{\sigma}(\mathcal{F}) &= \lim_{\longleftarrow m} A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m\mathcal{F})). \end{aligned}$$

By induction hypothesis applied to the scheme Y_{η_0} we can suppose that for any locally free sheaf \mathcal{H} on Y_{η_0} the map

$$0 \longrightarrow A_{\partial_{i-1}(\partial_0(\sigma))}(\mathcal{H}) \longrightarrow A_{\partial_0(\sigma)}(\mathcal{H}) \quad (43)$$

is an imbedding. Let us show that for any m the map

$$A_{\partial_{i-1}(\partial_0(\sigma))}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m\mathcal{F})) \longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m\mathcal{F})) \quad (44)$$

is an imbedding.

If $m = 1$, then, using (42), as in case 2, we reduce all at once to the sequence (43). Afterwards we pass to the direct limit.

If $m > 1$, then as in case 2 we can do induction on m by means of using of two following exact sequences:

$$\begin{aligned} 0 \longrightarrow A_{\partial_{i-1}(\partial_0(\sigma))}(j_{\eta_0})_*(j_{\eta_0})^*\mathcal{G} &\longrightarrow A_{\partial_{i-1}(\partial_0(\sigma))}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^m\mathcal{F}) \longrightarrow \\ &\longrightarrow A_{\partial_{i-1}(\partial_0(\sigma))}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^{m-1}\mathcal{F}) \longrightarrow 0 \\ 0 \longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{G}) &\longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^m\mathcal{F}) \longrightarrow \\ &\longrightarrow A_{\partial_0(\sigma)}((j_{\eta_0})_*(j_{\eta_0})^*\mathcal{F}/J_{\eta_0}^{m-1}\mathcal{F}) \longrightarrow 0, \end{aligned}$$

where the sheaf $\mathcal{G} = J_{\eta_0}^{m-1}\mathcal{F}/J_{\eta_0}^m\mathcal{F}$.

Now after passage in (44) to the limit we conclude the proof of case 3.

Now consider item 1 of the theorem. If $k = 0$, then this is corollary of lemma 12. If $k > 0$, then consider the map

$$H^0(X, \mathcal{F}) \longrightarrow A_\sigma(\mathcal{F})$$

as composition of the maps

$$H^0(X, \mathcal{F}) \longrightarrow A_{\partial_0(\sigma)}(\mathcal{F}) \quad \text{and} \quad A_{\partial_0(\sigma)}(\mathcal{F}) \longrightarrow A_\sigma(\mathcal{F}),$$

where we can suppose that by induction on k the first map is injective, and by item 2 of this theorem the second map is injective as well. Theorem 4 is proved.

Lemma 13 *Let all the conditions of theorem 4 be satisfied. By J_i denote the ideal sheaves on X defining the corresponding Y_i in X . Then for any locally free sheaf \mathcal{F} on X , for any $m > 0$ we have that*

1. the map

$$H^0(X, \mathcal{F}/J_i^m \mathcal{F}) \longrightarrow A_\sigma(\mathcal{F}/J_i^m \mathcal{F})$$

is an imbedding for any $\sigma = (\zeta_0, \dots, \zeta_k)$, $0 \leq i \leq \zeta_0$.

2. the map

$$A_{\sigma_1}(\mathcal{F}/J_i^m \mathcal{F}) \longrightarrow A_{\sigma_2}(\mathcal{F}/J_i^m \mathcal{F})$$

is an imbedding for any $\sigma_1, \sigma_2 \in S$, $\sigma_1 \subset \sigma_2 = (\eta_0, \dots, \eta_k)$, $0 \leq i \leq \eta_0$.

Proof. Let us show item 1. If $m = 1$, then this follows from theorem 4, which is applied to the scheme Y_i .

If $m > 1$, then apply induction. For this consider the exact sequence

$$0 \longrightarrow J_i^{m-1} \mathcal{F}/J_i^m \mathcal{F} \longrightarrow \mathcal{F}/J_i^m \mathcal{F} \longrightarrow \mathcal{F}/J_i^{m-1} \mathcal{F} \longrightarrow 0.$$

Hence and from exactness of the functor A_σ we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(X, J_i^{m-1} \mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{F}/J_i^{m-1} \mathcal{F}) & \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 0 \longrightarrow & A_\sigma(J_i^{m-1} \mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & A_\sigma(\mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & A_\sigma(\mathcal{F}/J_i^{m-1} \mathcal{F}) & \longrightarrow 0. \end{array}$$

The sheaf $J_i^{m-1} \mathcal{F}/J_i^m \mathcal{F} = \frac{J_i^{m-1}}{J_i^m} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free on Y_i (see [9, ch. II, th. 8.21A]). Therefore the map α is an imbedding. The map γ is an imbedding by the induction hypothesis. Now from non complicated diagram search it follows that β is an imbedding as well. Item 1 is proved.

Item 2 follows at once from analogous to item 1 and inductive on m reasonings applied to the following diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & A_{\sigma_1}(J_i^{m-1} \mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & A_{\sigma_1}(\mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & A_{\sigma_1}(\mathcal{F}/J_i^{m-1} \mathcal{F}) & \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 0 \longrightarrow & A_{\sigma_2}(J_i^{m-1} \mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & A_{\sigma_2}(\mathcal{F}/J_i^m \mathcal{F}) & \longrightarrow & A_{\sigma_2}(\mathcal{F}/J_i^{m-1} \mathcal{F}) & \longrightarrow 0. \end{array}$$

The lemma is proved.

Theorem 5 *Let all the conditions of theorem 4 be satisfied. Then for any locally free sheaf \mathcal{F} , for any $\sigma_1, \sigma_2 \in S$ we have that*

1. *if $\sigma_1 \cap \sigma_2 = \emptyset$, then*

$$A_{\sigma_1}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}) = H^0(X, \mathcal{F});$$

2. *if $\sigma_1 \cap \sigma_2 \neq \emptyset$, then*

$$A_{\sigma_1}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}) = A_{\sigma_1 \cap \sigma_2}(\mathcal{F}).$$

Remark 8 According to theorem 4, the intersections make sense, because we can always imbed $A_{\sigma_1}(\mathcal{F})$ and $A_{\sigma_2}(\mathcal{F})$ into $A_\eta(\mathcal{F})$, where η contains σ_1 and σ_2 . For instance, $\eta = \sigma_1 \cup \sigma_2$.

Proof. As usually, by J_i denote the ideal sheaves on X defining the corresponding subschemes Y_i in X .

Let us show item 1. Let $\sigma_1 = (\eta_0, \dots)$, $\sigma_2 = (\zeta_0, \dots)$, $\sigma_1 \cap \sigma_2 = \emptyset$. Without loss of generality it can be assumed that $\zeta_0 > \eta_0$. Assume that $\sigma_1 \notin S_0$.

By lemma 11 for any $m > 1$ we have the exact sequence:

$$0 \longrightarrow \mathcal{F}/J_{\eta_0}^m \mathcal{F} \longrightarrow (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \longrightarrow \mathcal{G}_m \longrightarrow 0,$$

where the sheaf $\mathcal{G}_m = \frac{(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})}{\mathcal{F}/J_{\eta_0}^m \mathcal{F}}$.

Let us show by induction on m that the natural map

$$A_{\partial_0(\sigma_1)}(\mathcal{G}_m) \longrightarrow A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{G}_m) \quad (45)$$

is an imbedding.

If $m = 1$, then

$$\begin{aligned} \mathcal{G}_1 &= \frac{(j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0} \mathcal{F})}{\mathcal{F}/J_{\eta_0} \mathcal{F}} = \frac{(j_{\eta_0})_*(j_{\eta_0})^* \mathcal{O}_{Y_{\eta_0}} \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{F}/J_{\eta_0} \mathcal{F})}{\mathcal{F}/J_{\eta_0} \mathcal{F}} = \\ &= \frac{\varinjlim_k \mathcal{O}(kY_{\eta_0+1}) \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{F}/J_{\eta_0} \mathcal{F})}{\mathcal{F}/J_{\eta_0} \mathcal{F}} = \varinjlim_{k>0} ((\mathcal{F}/J_{\eta_0} \mathcal{F}) \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{O}_{Y_{\eta_0}}(kY_{\eta_0+1})/\mathcal{O}_{Y_{\eta_0}})). \end{aligned}$$

Denote the sheaf $\mathcal{H}_k = (\mathcal{F}/J_{\eta_0} \mathcal{F}) \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{O}_{Y_{\eta_0}}(kY_{\eta_0+1})/\mathcal{O}_{Y_{\eta_0}})$. By induction on k let us show that the maps

$$A_{\partial_0(\sigma_1)}(\mathcal{H}_k) \longrightarrow A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{H}_k) \quad (46)$$

are imbeddings.

If $k = 1$, then the sheaf \mathcal{H}_1 is locally free on Y_{η_0+1} . Therefore in this case (46) follows from theorem 4 applied to Y_{η_0+1} .

If $k > 1$, then from the exact sequence

$$0 \longrightarrow \frac{\mathcal{O}_{Y_{\eta_0}}((k-1)Y_{\eta_0+1})}{\mathcal{O}_{Y_{\eta_0}}} \longrightarrow \frac{\mathcal{O}_{Y_{\eta_0}}(kY_{\eta_0+1})}{\mathcal{O}_{Y_{\eta_0}}} \longrightarrow \frac{\mathcal{O}_{Y_{\eta_0}}(kY_{\eta_0+1})}{\mathcal{O}_{Y_{\eta_0}}((k-1)Y_{\eta_0+1})} \longrightarrow 0$$

it follows the commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & A_{\partial_0(\sigma_1)}(\mathcal{H}_{k-1}) & \longrightarrow & A_{\partial_0(\sigma_1)}(\mathcal{H}_k) & \longrightarrow & A_{\partial_0(\sigma_1)}(\mathcal{H}_k/\mathcal{H}_{k-1}) & \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 \longrightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{H}_{k-1}) & \longrightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{H}_k) & \longrightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{H}_k/\mathcal{H}_{k-1}) & \longrightarrow 0.
\end{array}$$

The sheaf $\mathcal{H}_k/\mathcal{H}_{k-1} = (\mathcal{F}/J_{\eta_0}\mathcal{F}) \otimes_{\mathcal{O}_{Y_{\eta_0}}} (\mathcal{O}_{Y_{\eta_0}}(kY_{\eta_0+1})/\mathcal{O}_{Y_{\eta_0}}((k-1)Y_{\eta_0+1}))$ is locally free on Y_{η_0+1} . Therefore the map γ is injective by theorem 4 applied to Y_{η_0+1} . The map α is injective by induction hypothesis. Hence we have that the map β is injective as well. Thus (46) is proved. After passage in (46) to the direct limit on k we obtain (45) in the case $m = 1$.

Now let us show (45) in the case $m > 1$. From the exact sequences:

$$0 \longrightarrow \frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}} \longrightarrow \mathcal{F}/J_{\eta_0}^m\mathcal{F} \longrightarrow \mathcal{F}/J_{\eta_0}^{m-1}\mathcal{F} \longrightarrow 0$$

and

$$0 \longrightarrow (j_{\eta_0})_*(j_{\eta_0})^*\left(\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}\right) \longrightarrow (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m\mathcal{F}) \longrightarrow (j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^{m-1}\mathcal{F}) \longrightarrow 0$$

it follows the exact sequence

$$0 \longrightarrow \frac{(j_{\eta_0})_*(j_{\eta_0})^*\left(\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}\right)}{\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}} \longrightarrow \mathcal{G}_m \longrightarrow \mathcal{G}_{m-1} \longrightarrow 0.$$

Applying the exact functors $A_{\partial_0(\sigma_1)}$ and $A_{\partial_0(\sigma_1) \cup \sigma_2}$, we obtain the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & A_{\partial_0(\sigma_1)}\left(\frac{(j_{\eta_0})_*(j_{\eta_0})^*\left(\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}\right)}{\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}}\right) & \longrightarrow & A_{\partial_0(\sigma_1)}(\mathcal{G}_m) & \longrightarrow & A_{\partial_0(\sigma_1)}(\mathcal{G}_{m-1}) & \longrightarrow 0 \\
& \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\
0 \longrightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}\left(\frac{(j_{\eta_0})_*(j_{\eta_0})^*\left(\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}\right)}{\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}}}\right) & \longrightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{G}_m) & \longrightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{G}_{m-1}) & \longrightarrow 0.
\end{array}$$

The sheaf $\frac{J_{\eta_0}^{m-1}\mathcal{F}}{J_{\eta_0}^m\mathcal{F}} = \frac{J_{\eta_0}^{m-1}}{J_{\eta_0}^m} \otimes_{Y_{\eta_0}} (\mathcal{F}/J_{\eta_0}\mathcal{F})$ is locally free on Y_{η_0} . Therefore the map α is injective by the same reasons as in the case $m = 1$. The map γ is injective by the inductive hypothesis. Therefore the map β is injective as well (this follows from non complicated diagram search). Thus we have shown (45).

Now consider the following diagram.

$$\begin{array}{ccccccc}
0 \rightarrow A_{\partial_0(\sigma_1)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) & \rightarrow & A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) & \xrightarrow{\psi} & A_{\partial_0(\sigma_1)}(\mathcal{G}_m) & \rightarrow & 0 \\
\downarrow \phi & & \downarrow \theta & & \downarrow \beta & & \\
0 \rightarrow A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) & \rightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) & \rightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{G}_m) & \rightarrow & 0.
\end{array}$$

(Note also that here the map θ is injective. This follows from the fact that the map β is injective by the above, and the statement ϕ is injective by lemma 13.)

Now let an element

$$x \in A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})), \quad \text{but}$$

$$x \notin A_{\partial_0(\sigma_1)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}).$$

Then since the map β is injective, we have that

$$\beta\psi(x) \neq 0. \tag{47}$$

Now consider the diagram

$$\begin{array}{ccccccc}
0 \rightarrow A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) & \rightarrow & A_{\sigma_2}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) & \xrightarrow{\psi_1} & A_{\sigma_2}(\mathcal{G}_m) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \beta_1 & & \\
0 \rightarrow A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) & \rightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) & \rightarrow & A_{\partial_0(\sigma_1) \cup \sigma_2}(\mathcal{G}_m) & \rightarrow & 0.
\end{array}$$

(Similarly to the previous reasonings we have that in this diagram all the vertical arrows are injective.)

And if an element

$$x \in A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}), \quad \text{then}$$

$$\beta_1\psi_1(x) = 0. \tag{48}$$

Now if

$$x \in A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \cap A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F})$$

(where the intersection is possible to be taken in $A_{\partial_0(\sigma_1) \cup \sigma_2}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F}))$, because all the appearing maps are injective), then from functoriality we have

$$\beta\psi(x) = \beta_1\psi_1(x).$$

Therefore, comparing this with (47) and (48), we obtain that in

$A_{\partial_0(\sigma_1) \cup \sigma_2}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F}))$ is satisfied

$$A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \cap A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) = A_{\partial_0(\sigma_1)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}).$$

Now from the definition of A_σ we have that

$$A_{\sigma_1}(\mathcal{F}) = \varprojlim_m A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})),$$

from lemma 10 we have that

$$A_{\sigma_2}(\mathcal{F}) = \varprojlim_m A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}).$$

Therefore,

$$\begin{aligned} A_{\sigma_1}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}) &= \left(\varprojlim_m A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \right) \cap \left(\varprojlim_m A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \right) = \\ &= \varprojlim_m \left(A_{\partial_0(\sigma_1)}((j_{\eta_0})_*(j_{\eta_0})^*(\mathcal{F}/J_{\eta_0}^m \mathcal{F})) \cap A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \right) = \\ &= \varprojlim_m \left(A_{\partial_0(\sigma_1)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \right) = \\ &= \left(\varprojlim_m A_{\partial_0(\sigma_1)}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \right) \cap \left(\varprojlim_m A_{\sigma_2}(\mathcal{F}/J_{\eta_0}^m \mathcal{F}) \right) = A_{\partial_0(\sigma_1)}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}). \end{aligned}$$

Acting further in this manner, i. e., eliminating the minimal number in the union of indices every time, we obtain that

$$A_{\sigma_1}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}) = A_{(i)}(\mathcal{F}) \cap A_{\sigma}(\mathcal{F}), \quad \text{where}$$

$\sigma = (\zeta_0, \dots)$ and $\zeta_0 > i$. But in this case by the reasonings, which is completely analogous to the above, we obtain at once that

$$\begin{aligned} H^0(X, (j_i)_*(j_i)^*(\mathcal{F}/J_i^m \mathcal{F})) \cap A_{\sigma}(\mathcal{F}/J_i^m \mathcal{F}) &= \\ = H^0(X, \mathcal{F}/J_i^m \mathcal{F}) \cap A_{\sigma}(\mathcal{F}/J_i^m \mathcal{F}) &= H^0(X, \mathcal{F}/J_i^m \mathcal{F}). \end{aligned}$$

(Note that in contrast to the reasonings above with the functor A_{γ} , the functor $H^0(X, \cdot)$ is a left exact functor only. But the key diagram works in this case as well:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3. \end{array}$$

If the maps α and γ are injective, then the map β is injective as well.)

Now

$$\begin{aligned} A_i(\mathcal{F}) \cap A_{\sigma}(\mathcal{F}) &= \left(\varprojlim_m H^0(X, (j_i)_*(j_i)^*(\mathcal{F}/J_i^m \mathcal{F})) \right) \cap \left(\varprojlim_m A_{\sigma}(\mathcal{F}/J_i^m \mathcal{F}) \right) = \\ &= \varprojlim_m \left(H^0(X, (j_i)_*(j_i)^*(\mathcal{F}/J_i^m \mathcal{F})) \cap A_{\sigma}(\mathcal{F}/J_i^m \mathcal{F}) \right) = \varprojlim_m H^0(X, \mathcal{F}/J_i^m \mathcal{F}) = H^0(X, \mathcal{F}), \end{aligned}$$

where the last equality follows from lemma 12. Item 1 of theorem 5 is proved.

Now let us show item 2 of the theorem. Consider a few cases.

Case 1. $\sigma_1 \cap \sigma_2 \neq \emptyset$, $0 \notin \sigma_1$, $0 \notin \sigma_2$.

By lemma 10 we have that

$$\begin{aligned} A_{\sigma_1}(\mathcal{F}) &= \varprojlim_m A_{\sigma_1}(\mathcal{F}/J_1^m \mathcal{F}), & A_{\sigma_2}(\mathcal{F}) &= \varprojlim_m A_{\sigma_2}(\mathcal{F}/J_1^m \mathcal{F}), \\ A_{\sigma_1 \cap \sigma_2}(\mathcal{F}) &= \varprojlim_m A_{\sigma_1 \cap \sigma_2}(\mathcal{F}/J_1^m \mathcal{F}). \end{aligned}$$

Let us show that for any $m \geq 1$

$$A_{\sigma_1 \cap \sigma_2}(\mathcal{F}/J_1^m \mathcal{F}) = A_{\sigma_1}(\mathcal{F}/J_1^m \mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}/J_1^m \mathcal{F}), \quad (49)$$

where the last intersection is regarded in $A_{\sigma_1 \cup \sigma_2}(\mathcal{F}/J_1^m \mathcal{F})$. (By lemma 13 we can imbed these groups there.)

Let us prove (49) by induction on m . Let $m = 1$. In this case $\mathcal{F}/J_1 \mathcal{F}$ is a locally free sheaf on Y_1 . And equality (49) turns into the analogous equality (49) on Y_1 . The scheme Y_1 has lesser dimension than dimension of X . Applying induction on dimension of scheme, we can suppose that theorem 5 is already true for schemes of lesser dimension. (For schemes of dimension 1 theorem 5 follows at once from theorem 4 and theorem 3 by trivial reasons.) Therefore (49) is true when $m = 1$.

Let $m > 1$. Then the exact sequence

$$0 \longrightarrow J_1^{m-1} \mathcal{F}/J_1^m \mathcal{F} \longrightarrow \mathcal{F}/J_1^m \mathcal{F} \longrightarrow \mathcal{F}/J_1^{m-1} \mathcal{F} \longrightarrow 0$$

induces the following commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow A_{\sigma_1 \cap \sigma_2}(J_1^{m-1} \mathcal{F}/J_1^m \mathcal{F}) & \rightarrow & A_{\sigma_1 \cap \sigma_2}(\mathcal{F}/J_1^m \mathcal{F}) & \rightarrow & A_{\sigma_1 \cap \sigma_2}(\mathcal{F}/J_1^{m-1} \mathcal{F}) & \rightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 \rightarrow A_{\sigma_1}(\frac{J_1^{m-1} \mathcal{F}}{J_1^m \mathcal{F}}) \cap A_{\sigma_2}(\frac{J_1^{m-1} \mathcal{F}}{J_1^m \mathcal{F}}) & \rightarrow & H_m & \rightarrow & \frac{H_m}{A_{\sigma_1}(\frac{J_1^{m-1} \mathcal{F}}{J_1^m \mathcal{F}}) \cap A_{\sigma_2}(\frac{J_1^{m-1} \mathcal{F}}{J_1^m \mathcal{F}})} & \rightarrow & 0, \end{array}$$

where $H_m = A_{\sigma_1}(\mathcal{F}/J_1^m \mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}/J_1^m \mathcal{F})$. From $\sigma_1 \cap \sigma_2 \neq \emptyset$ it follows that the functor $A_{\sigma_1 \cap \sigma_2}$ is exact. Therefore the upper row of the diagram is exact.

There is the natural map θ :

$$\frac{H_m}{A_{\sigma_1}(\frac{J_1^{m-1} \mathcal{F}}{J_1^m \mathcal{F}}) \cap A_{\sigma_2}(\frac{J_1^{m-1} \mathcal{F}}{J_1^m \mathcal{F}})} \longrightarrow H_{m-1}.$$

And from the exact sequences

$$\begin{aligned} 0 \longrightarrow A_{\sigma_1}(J_1^{m-1} \mathcal{F}/J_1^m \mathcal{F}) &\longrightarrow A_{\sigma_1}(\mathcal{F}/J_1^m \mathcal{F}) \longrightarrow A_{\sigma_1}(\mathcal{F}/J_1^{m-1} \mathcal{F}) \\ 0 \longrightarrow A_{\sigma_2}(J_1^{m-1} \mathcal{F}/J_1^m \mathcal{F}) &\longrightarrow A_{\sigma_2}(\mathcal{F}/J_1^m \mathcal{F}) \longrightarrow A_{\sigma_2}(\mathcal{F}/J_1^{m-1} \mathcal{F}) \end{aligned}$$

it follows at once that the map θ is an imbedding. Besides, the map $\theta \cdot \gamma$ is the natural map from $A_{\sigma_1 \cap \sigma_2}(\mathcal{F}/J_1^{m-1}\mathcal{F})$ to H_{m-1} ; and, consequently, by the induction hypothesis it is possible to suppose that $\theta \cdot \gamma$ is an isomorphism.

From the last two facts we obtain at once that γ is an isomorphism. Since the sheaf $J_1^{m-1}\mathcal{F}/J_1^m\mathcal{F} = J_1^{m-1}/J_1^m \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free on Y_1 and $\dim Y_1 < \dim X$,

we have that the map α is an isomorphism as well. Therefore from this commutative diagram it follows that the map β is an isomorphism as well.

Thus, equality (49) is proved. Now passage in (49) to the projective limit on m concludes the proof of case 1.

Case 2. $0 \in \sigma_1$, $0 \notin \sigma_2$ (or vice versa), $\sigma_1 \cap \sigma_2 \neq \emptyset$.

Now by the analogous reasonings, as in the proof of item 1 of this theorem, we obtain at once the following

$$A_{\sigma_1}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}) = A_{\partial_0(\sigma_1)}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F});$$

that reduces this case to the case 1, analyzed above.

Case 3. $0 \in \sigma_1$, $0 \in \sigma_2$.

Then

$$A_{\sigma_1}(\mathcal{F}) = \varinjlim_k A_{\partial_0(\sigma_1)}(\mathcal{F}(kY_1)), \quad A_{\sigma_2}(\mathcal{F}) = \varinjlim_k A_{\partial_0(\sigma_2)}(\mathcal{F}(kY_1))$$

Now from case 1 (or if $\partial_0(\sigma_1) \cap \partial_0(\sigma_2) = \emptyset$, then from item 1 of this theorem) we have that

$$A_{\partial_0(\sigma_1)}(\mathcal{F}(kY_1)) \cap A_{\partial_0(\sigma_2)}(\mathcal{F}(kY_1)) = A_{\partial_0(\sigma_1) \cap \partial_0(\sigma_2)}(\mathcal{F}(kY_1)).$$

(Here $A_{\emptyset}(\cdot) = H^0(X, \cdot)$).

Therefore,

$$\begin{aligned} A_{\sigma_1}(\mathcal{F}) \cap A_{\sigma_2}(\mathcal{F}) &= \left(\varinjlim_k A_{\partial_0(\sigma_1)}(\mathcal{F}(kY_1)) \right) \cap \left(\varinjlim_k A_{\partial_0(\sigma_2)}(\mathcal{F}(kY_1)) \right) = \\ &= \varinjlim_k \left(A_{\partial_0(\sigma_1)}(\mathcal{F}(kY_1)) \cap A_{\partial_0(\sigma_2)}(\mathcal{F}(kY_1)) \right) = \varinjlim_k A_{\partial_0(\sigma_1) \cap \partial_0(\sigma_2)}(\mathcal{F}(kY_1)) = A_{\sigma_1 \cap \sigma_2}(\mathcal{F}). \end{aligned}$$

Theorem 5 is proved.

In the sequel we shall assume that all the conditions of theorem 4 are satisfied, and a field k is the field of definition of the scheme X . Also, let us assume that $Y_n = x$, where x is a k -rational point on X which is smooth on any Y_i ($0 \leq i \leq n$). Let us choose and fix local parameters $z_1, \dots, z_n \in \hat{\mathcal{O}}_{x,X}$ such that $z_{n-i+1}|_{Y_{i-1}} = 0$ is a local equation of the divisor Y_i in the formal neighbourhood of the point x on the scheme Y_{i-1} ($1 \leq i \leq n$). Let \mathcal{F} be a rank 1 locally free sheaf on X . Fix a trivialization e_x of the sheaf \mathcal{F} in the formal neighbourhood of the point x on X . Now the done choice of local parameters and trivialization makes possible to identify $A_{(0,1,\dots,n)}(\mathcal{F})$ with the n -dimensional local field $k((z_1)) \dots ((z_n))$.

Moreover, let us fix a collection of integers $0 \leq j_1 \leq \dots \leq j_k \leq n-1$. Define $\sigma \in S_{n-k}$ as the set $\{i : 0 \leq i \leq n, i \neq j_1, \dots, i \neq j_k\}$. By theorem 4 we have the natural imbedding $A_\sigma(\mathcal{F}) \longrightarrow A_{(0,1,\dots,n)}(\mathcal{F})$. And under identifying of $A_{(0,1,\dots,n)}(\mathcal{F})$ with the field $k((z_1)) \dots ((z_n))$ the space $A_\sigma(\mathcal{F})$ converts to the following k -subspace in $k((z_1)) \dots ((z_n))$:

$$\left\{ \sum a_{i_1, \dots, i_n} z_1^{i_1} \dots z_n^{i_n} : a_{i_1, \dots, i_n} \in k, i_{n-j_1} \geq 0, i_{n-j_2} \geq 0, \dots, i_{n-j_k} \geq 0 \right\}. \quad (50)$$

Thus, from theorem 5 we obtain that for determination of the images of $A_\sigma(\mathcal{F})$ in $k((z_1)) \dots ((z_n))$ (for any $\sigma \in S$) it suffices to know only one image of $A_{(0,1,\dots,n-1)}$ in $k((z_1)) \dots ((z_n))$. (All the others are obtained by intersection of the image of $A_{(0,1,\dots,n-1)}$ in $k((z_1)) \dots ((z_n))$ with the standard subspaces (50) in $k((z_1)) \dots ((z_n))$.)

It is clear that these reasonings is generalized at once to locally free sheaves \mathcal{F} of rank r and spaces $k((z_1)) \dots ((z_n))^{\oplus r}$.

Denote the described map

$$(X, Y_1, \dots, Y_n, (z_1, \dots, z_n), \mathcal{F}, e_x) \longrightarrow A_{(0,1,\dots,n-1)}(\mathcal{F}) \hookrightarrow A_{(0,1,\dots,n-1,n)}(\mathcal{F}) \xrightarrow{e_x} \\ \xrightarrow{e_x} A_{(0,1,\dots,n)}(\hat{\mathcal{O}}_{x,X}) \xrightarrow{z_1, \dots, z_n} k((z_1)) \dots ((z_n))^{\oplus r}$$

by Ψ_r .

Definition.

\mathcal{M}_n	$\stackrel{\text{def}}{=} \{X, (Y_1, \dots, Y_n), (z_1, \dots, z_n), \mathcal{F}, e_{Y_n}\}$
X	<i>a projective equidimensional Cohen-Macaulay scheme of dimension n over a field k</i>
$X = Y_0 \supset Y_1 \supset \dots \supset Y_n$	<i>a flag of closed subschemes such that Y_i is an ample Cartier divisor on the scheme Y_{i-1} ($1 \leq i \leq n$)</i>
Y_n	<i>a smooth k-rational point on all Y_i ($0 \leq i \leq n$)</i>
z_1, \dots, z_n	<i>formal local parameters in the point Y_n such that $(z_{n-i+1} _{Y_{i-1}} = 0) = Y_i$ in the formal neighbourhood of the point Y_n on the scheme Y_{i-1}</i>
\mathcal{F}	<i>a rank r vector bundle on X</i>
e_{Y_n}	<i>a trivialization of \mathcal{F} in the formal neighbourhood of the point Y_n on X</i>

In the field $K = k((z_1)) \dots ((z_n))$ we have the following filtration

$$K(m) = z_n^m k((z_1)) \dots ((z_{n-1}))[[z_n]].$$

Let K -space $V = K^{\oplus r}$, and let the filtration $V(m) = K(m)^{\oplus r}$.

Theorem 6 *There exists a canonical map*

$$\Phi_n : \mathcal{M}_n \longrightarrow \{k\text{-vector spaces } B \subset K, \quad W \subset V\}$$

such that

1. *from the subspace $B \subset K$ is uniquely reconstructed the complex $A(\mathcal{O}_X)$, which calculates cohomology of the sheaf \mathcal{O}_X on X ;*
2. *from the subspace $W \subset V$ is uniquely reconstructed the complex $A(\mathcal{F})$, which calculates cohomology of the sheaf \mathcal{F} on X ;*
3. *if $(B, W) \in \text{Im } \Phi_n$, then $B \cdot B \subset B$, $B \cdot W \subset W$;*
4. *for all m the map*

$$\begin{aligned} & \{Y_1, (Y_2, \dots, Y_n), (z_1, \dots, z_{n-1})|_{Y_1}, \mathcal{F}(-mY_1)|_{Y_1}, e_{Y_n}(-m)|_{Y_1}\} \longrightarrow \\ & \longrightarrow \left\{ \frac{B \cap K(m)}{B \cap K(m+1)} \subset \frac{K(m)}{K(m+1)} = k((z_1)) \dots ((z_{n-1})), \right. \\ & \quad \left. \frac{W \cap V(m)}{W \cap V(m+1)} \subset \frac{V(m)}{V(m+1)} = k((z_1)) \dots ((z_{n-1}))^{\oplus r} \right\} \end{aligned}$$

coincides with the map Φ_{n-1} ;

5. *If $q, q' \in \mathcal{M}_n$ and $\Phi_n(q) = \Phi_n(q')$, then q is isomorphic to q' .*

Proof. If

$$q = \{X, (Y_1, \dots, Y_n), (z_1, \dots, z_n), \mathcal{F}, e_{Y_n}\} \in \mathcal{M}_n,$$

then to define the map Φ_n we put

$$B = \Psi_1(X, Y_1, \dots, Y_n, (z_1, \dots, z_n), \mathcal{O}_X, id),$$

$$W = \Psi_r(X, Y_1, \dots, Y_n, (z_1, \dots, z_n), \mathcal{F}, e_{Y_n}),$$

$$\Phi_n(q) = \{B, W\}.$$

Now items 1-4 of this theorem follows from theorems 3, 4, 5 and reasonings above about the map Ψ , and also for item 4 is needed the fact that $(j_0)_*(j_0)^*\mathcal{F} = \varinjlim_m \mathcal{F}(mY_1)$.

Let us show item 5. Intersecting B with the standard subspaces (50), we can uniquely reconstruct the algebra $A_{(0)}(\mathcal{O}_X) \subset K$. Similarly, from W we can reconstruct the k -subspace $A_{(0)}(\mathcal{F}) \subset V$. Then

$$X - Y_1 = \text{Spec } A_{(0)}(\mathcal{O}_X)$$

$$X = \text{Proj} \left(\bigoplus_{m \geq 0} (A_{(0)}(\mathcal{O}_X) \cap K(-m)) \right) \quad (51)$$

$$\mathcal{F} = \text{Proj} \left(\bigoplus_{m \geq 0} (A_{(0)}(\mathcal{F}) \cap V(-m)) \right),$$

where the last equalities follow from the following statement (see [17, lemma 7]): if X is a projective scheme over a field, \mathcal{F} is a coherent sheaf on X , and C is an ample divisor on X , then $X \cong \text{Proj}(S)$, $\mathcal{F} \cong \text{Proj}(F)$, where $S = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mC))$, $F = \bigoplus_{m \geq 0} H^0(X, \mathcal{F}(mC))$.

Besides, the image under the imbedding of

$$\bigoplus_{m \geq 0} (A_{(0)}(\mathcal{O}_X) \cap K(-m+1)) \quad \text{into} \quad \bigoplus_{m \geq 0} (A_{(0)}(\mathcal{O}_X) \cap K(-m))$$

is the homogeneous ideal determining the subscheme Y_1 in X . Now, using item 4 of this theorem, we can reconstruct all the geometrical data on the subscheme Y_1 in the analogous way, and, further, by induction we can reconstruct all the data from q up to an isomorphism. Theorem 6 is proved.

Remark 9 Note that $\mathcal{F}(nY_1)|_{Y_1} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(nY_1)|_{Y_1}$ and the sheaf

$$\mathcal{O}(nY_1)|_{Y_1} = \mathcal{O}(nY_1)/\mathcal{O}((n-1)Y_1) = \mathcal{N}_{Y_1/X}^n,$$

where the bundle $\mathcal{N}_{Y_1/X}$ coincides with the normal bundle of Y_1 in X in some cases (for example, if X and Y_1 are smooth).

Remark 10 From (51) and absence of divisors of zero in the field K it follows at once that the schemes X, Y_1, \dots, Y_n , satisfied the conditions of the definition \mathcal{M}_n , are always integral schemes.

Remark 11 Φ_1 is a variant of the Krichever correspondence for curves (see [13], [17]). Besides, any integral noetherian scheme of dimension 1 is a Cohen-Macaulay scheme.

Φ_2 coincides with the map, constructed in [17]. Note that any normal noetherian scheme of dimension 2 is a Cohen-Macaulay scheme (see [9, ch. II, th. 8.22A]). Also in [17] is analyzed an example of the Krichever map for $X = \mathbf{P}_2$.

Remark 12 In [17] is discussed the problem of change of locally free sheaves to torsion free sheaves.

Let X be a smooth projective surface with a flag of irreducible subvarieties $Y_1 \supset Y_2$ such that Y_1 is an ample divisor on X , and Y_2 is a point. Let \mathcal{G} be any torsion free sheaf on X . Then we have the imbedding

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{**}. \quad (52)$$

The sheaf \mathcal{G}^{**} is reflexive; and since $\dim X = 2$, we have that \mathcal{G}^{**} is a locally free sheaf. Now applying the exact functors A_σ and the left exact functor $H^0(X, \cdot)$ to sequence (52), we obtain at once from the obtained sequences and theorem 4 for the sheaf \mathcal{G}^{**} that theorem 4 is true for the sheaf \mathcal{G} as well.

But theorem 5 is no longer true for torsion free sheaves on X . Indeed, let $\mathcal{G} = m_Q$, where m_Q is the ideal sheaf of a point $Q \in X$. Then, applying the exact functors A_σ to the exact sequence

$$0 \longrightarrow m_Q \longrightarrow \mathcal{O}_X \longrightarrow k_Q \longrightarrow 0$$

we obtain at once that

- if $Q \notin Y_1$, then

$$A_{(0)}(\mathcal{G}) \neq A_{(01)}(\mathcal{G}) \cap A_{(02)}(\mathcal{G}),$$

because $A_{(01)}(\mathcal{G}) \neq A_{(01)}(\mathcal{O}_X)$, but $A_{(01)}(\mathcal{G}) = A_{(01)}(\mathcal{O}_X)$, $A_{(02)}(\mathcal{G}) = A_{(02)}(\mathcal{O}_X)$;

- if $Q \in Y_1$, $Q \neq Y_2$, then

$$A_{(1)}(\mathcal{G}) \neq A_{(01)}(\mathcal{G}) \cap A_{(12)}(\mathcal{G}),$$

because $A_{(1)}(\mathcal{G}) \neq A_{(1)}(\mathcal{O}_X)$, but $A_{(01)}(\mathcal{G}) = A_{(01)}(\mathcal{O}_X)$, $A_{(12)}(\mathcal{G}) = A_{(12)}(\mathcal{O}_X)$.

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